Conjugacy in braid groups and the LKB representation; and Bessis-Garside groups of rank 3

by

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Contents

Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree. No parts of this thesis have been published by the author.

Abstract

In the first part of this thesis, we give a survey of the conjugacy problem in the braid group, describing the solution provided by Garside theory, and outlining the progress that has been made towards a polynomial time solution in recent years using refinements of Garside's solution, and the Thurston-Nielsen classification of braids, which reduces the problem to the case of pseudo-Anosov braids. Using the faithful Lawrence-Krammer-Bigelow representation of the braid groups, we consider how the eigenspaces of pseudo-Anosov braids can under certain conditions yield invariants of their conjugacy class and thus lead us towards a polynomial time solution of the conjugacy problem.

In the second part we introduce Bessis-Garside groups, a generalisation of the methods used by Bessis in his papers on dual braid monoids. We consider the groups given by taking the quotient of the free group by the orbits of its generators under the action of some subgroup of the braid group, and find that in many cases this construction can give us a group with a Garside structure. By means of introduction we review the simple rank 2 case, and summarise examples of such groups already known to admit Garside structures, in particular due to the work of Digne. We then go on to give all those of such groups which can be found as quotients of affine and spherical Artin groups of rank 3. We show that all such groups may be given a cycle presentation, or equivalently may be given as labelled-oriented-graph presented groups, and give conditions on such presentations that are equivalent to the group admitting a 'dual' Garside structure. Restricting by the cycle lengths occuring in such presentations we give all Bessis-Garside groups of rank 3 which have all cycles length at most 4, and discuss the case of Bessis-Garside groups with uniform cycle length.

Chapter 1

A survey of conjugacy in braid groups

1.1 Introduction

This thesis consists of two parts, in this, the first, we will consider the conjugacy problem in the braid group. In this chapter we lay the groundwork by giving an overview of the current state of research on this subject. In the following section 1.2 we will define the braid group on n strands, denoted B_n, discuss its interpretation in terms of geometric braids, and give the standard presentation originating with Artin [1].

In section 1.3 we introduce Garside structures, and in particular describe the classical Garside structure of the braid group, showing how it yields a solution to the word problem for these groups. We then go on, in section 1.4, to see how this approach also yields a solution to the conjugacy problem, using the notion of summit sets. For both of these results we follow Garside's original paper on the matter [23]. At this point we move on from Garside to see how two refinements of his method (namely super and ultra summit sets, due to Elrifrai and Morton [22] and Gebhardt [24] respectively) result in improved solutions to the conjugacy problem.

In the final section of this chapter 1.5, we outline the scheme given in [10], to find a polynomial solution to the conjugacy problem, by using the fact that the braid group on n strands is isomorphic to the mapping class group of the n-punctured disk. This fact allows us to use the Thurston-Nielsen classification to reduce the conjugacy problem to the case of pseudo-Anosov braids, and in indeed to only those pseudo-Anosov braids admitting the property of rigidity.

1.2 The Braid Group

1.2.1 Definition

The braid group (on n strands) has a number of equivalent definitions, we choose to define it as the fundamental group of the braid space of n braids.

Definition 1.2.1 *We define the **braid space of** n **braids**, BS_n, to be the configuration space of n unordered distinct points in the complex plane, that is:*

$$BS_n = \{(x_1, \dots, x_n) \mid x_i \in \mathbb{C} \ \forall \ i, x_i \neq x_j \ \text{for} \ i \neq j\} / \Sigma_n$$

where Σ_n is the permutation group on the entries x_i.

We then define the braid group as follows.

Definition 1.2.2 (The braid group) *The **braid group** B_n is the fundamental group of the braid space, that is:*

$$B_n := \pi_1(BS_n, X_0)$$

for some base point X_0.

The elements of B_n, called braids, are then isotopy classes of loops in BS_n centred at X_0

1.2.2 Geometric Braids

To understand why the above group is called the braid group and to allow us to visualise braids, we introduce geometric braids.

Definition 1.2.3 (Geometric Braids) *Let f be a smooth map from the unit interval to the braid space of n braids, that is:*

$$f : [0, 1] \to \{X \subset \mathbb{C} : |X| = n\}$$

*such that $f(0) = f(1)$ Then the **geometric braid** associated to f is:*

$$GB(f) := \bigcup_{t \in [0,1]} (f(t) \times \{t\})$$

So each geometric braid is the path traversed by a loop in the braid space, and therefore has associated a representative of a braid.

In order to draw geometric braids, we can introduce braid diagrams. The **braid diagram** of a geometric braid is given by projecting it onto the plane $\mathbb{R} \times [0, 1]$, $\mathbb{R}$ being the real line in $\mathbb{C}$, indicating for any intersections which strand passes in front of the other. As an example see Figure 1.1.

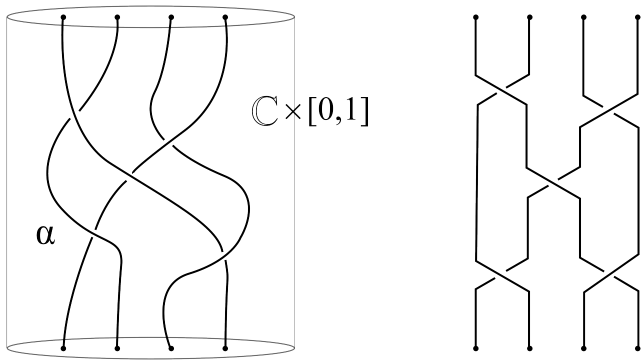

Figure 1.1: A geometric braid and its braid diagram.

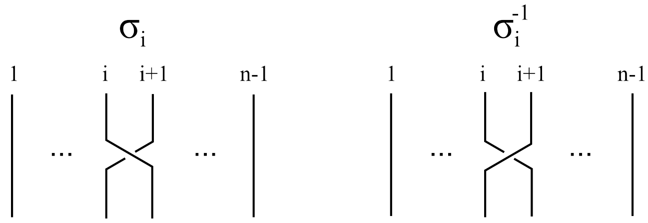

Figure 1.2: The Artin generators of the braid group.

So given any braid on n-strands, we can choose a representative geometric braid, and visualise it as a braid diagram. Conversely, each braid diagram gives us a choice of associated geometric braids, which all represent the same braid.

1.2.3 The Artin Presentation

From the diagrammatic viewpoint, we can easily see that the braid group is generated by the braids σ_i for $i = 1, \dots, n-1$, given by the i-th strand over the $i+1$-th strand, and all other strands remaining motionless; and their inverses σ_i^{-1} for $i = 1, \dots, n-1$ (see Figure 1.2).

Furthermore we can see that certain isotopies are represented by braid diagrams which recall the Reidemeister moves of knot theory, (see Figure 1.3). Specifically there is an isotopy between $\sigma_i\sigma_{i+1}\sigma_i$ and $\sigma_{i+1}\sigma_i\sigma_{i+1}$ which diagramatically involves the 3rd Reidemeister move; and between $\sigma_i\sigma_i^{-1}$ and the trivial braid (the 2nd Reidemeister move). Finally there is clearly an isotopy between $\sigma_i\sigma_j$ and $\sigma_j\sigma_i$ when $|i-j| \geq 2$.

In [1] Artin proved that these moves in fact represent all isotopies, so that a braid is determined by the braid diagram of a representative geometric braid, up to Reidemeister moves. Thus from the braid diagram viewpoint we get the following presentation of the braid group:

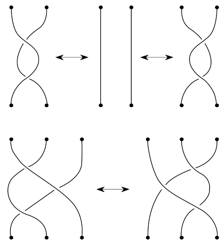

Figure 1.3: Some isotopies of braids.

Theorem 1.2.4 *The braid group B_n admits the following presentation:*

$$B_n \cong \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{r} \sigma_i \cdot \sigma_j = \sigma_j \cdot \sigma_i \ \textit{if}\ |i-j| \geq 2 \\ \sigma_i \cdot \sigma_j \cdot \sigma_i = \sigma_j \cdot \sigma_i \cdot \sigma_j \ \textit{if}\ |i-j| = 1 \end{array} \right\rangle$$

For more on the basics of braids see for example [26].

1.3 The Word Problem

The word problem is a standard decision problem in group theory: given a generating set for a group, and two words in those generators, how can we determine whether they give the same element? In the case of the braid group, given two braid words, in the generators σ_i and their inverses σ_i^{-1}, how can we determine whether the braids they represent are isotopic? In the rest of this section we will outline Garside's solution to this problem, in the process introducing the basic definitions of Garside theory that will be fundamental in the remainder of this thesis.

1.3.1 Garside Structure

The solution given by Garside in [23], relies upon studying a number of properties satisfied by the braid groups (and a number of other groups). These structures together form what is now referred to as a Garside structure on the group.

The required properties are as follows:

Definition 1.3.1 *Let G be a group, and Δ be an element of G (called the Garside element). (G, Δ) is a Garside pair if we have:*

1. *A lattice order on the group which is invariant under left multiplication: that is a partial ordering, $\preceq$, such that $a \preceq b$ implies $ca \preceq cb$; and for any two elements a, b there exists a unique meet (or greatest lower bound) and join (or least upper bound), $a \wedge b =$ (and $a \vee b$). Note that this determines a positive cone in the group, $P = \{p \mid 1 \preceq p\}$.*
2. *The positive cone P is preserved under conjugation by Δ: $P = \Delta^{-1} P \Delta$. The set $[1, \Delta] = \{s \mid 1 \preceq s \preceq \Delta\}$ generates the group. The elements of $[1, \Delta]$ are called simple elements of the group.*

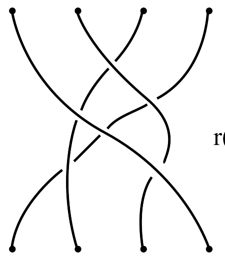

Figure 1.4: The map $r : \Sigma_n \to B_n$

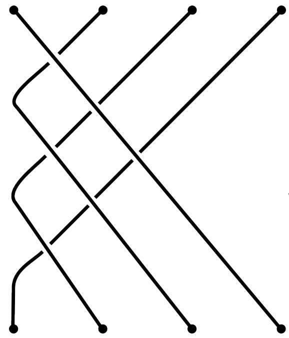

Figure 1.5: The Garside element, Δ.

3. *The positive cone is atomic. We define the atoms of P to be those elements in P which cannot be decomposed into a product of non-trivial elements in P. Then P is atomic if for any $x \in P$, there is an upper bound on the number of atoms in any decomposition of x into non-trivial atoms. Then for any $x \in P$, the length of x is the maximum length of such a decomposition $\|x\| = max\,\{n \mid x = a_1 a_2 \cdots a_n$ where a_i are atoms in $P - 1\}$.*

B_n in fact admits two Garside structures, the first of which was described by Garside, and is known as the classical Garside structure; the second was described by Birman, Ko, and Lee in [13] and is called the dual Garside structure. Our focus will be (for the moment) on the first of these. In order to describe the classical Garside structure on B_n, we consider a map $r : \Sigma_n \to B_n$, from the symmetric group to the braid group. This maps a permutation to the braid which has the same permutation of endpoints and contains only positive crossings, that is only positive powers of the generators, and in which any pair of strands crosses at most once, (see Figure 1.4).

If we consider the symmetric group to be generated by transpositions $(i, i+1)$, which are mapped by r to the Artin generators of the braid group, then we have a notion of canonical length in the symmetric group. This allows us to determine a longest element. The image of this element under r, which we denote Δ, will be the Garside element of classical Garside structure on the braid group (see Figure 1.5).

The partial ordering is given by left divisibility, that is $a \preceq b$ if there exists c such that $ac = b$, and the positive cone consists of the positive braids, and we denote it B_n^+.

The set $[1, \Delta] = r(\Sigma_n)$, is the set of simple elements.

The atoms of P are the Artin generators.

1.3.2 Solution to the Word Problem

Now that we have a Garside structure it remains to see how this leads to a solution of the word problem.

We first define the positive braids to be $B_n^+ := \langle \sigma_1, \ldots, \sigma_{n-1} \rangle \subset B_n$, that is the elements of the braid group that can be written as words in the generators only, without their inverses.

Suppose we have a word in B_n^+ written in terms of simple elements, $x = x_1, \ldots, x_n$, then considering each pair x_i, x_{i+1}, we have a rewriting system given by taking all crossings from x_{i+1} which aren't present in x_i across. Algebraically this consists of taking the pair x_i, x_{i+1}, and replacing it by $x_i \cdot s, \overline{x_{i+1}}$, where s and $\overline{x_{i+1}}$ are defined by $s = \tilde{x_i} \wedge x_{i+1}$ (where $\tilde{w}$ is the word w over the braid generators with the order of those generators reversed) and $x_{i+1} = s \cdot \overline{x_{i+1}}$. This is in fact a complete rewriting system and thus every positive braid word can be written uniquely in the following form:

$$\Delta^p y_1 \cdots y_r$$

where p is some integer, each y_i is a simple braid and $y_1 \neq \Delta$.

If x is a word in $B_n \setminus B_n^+$ we must first use a process called word reversing to rewrite x in the form $v^{-1}w$ where both v and w are in B_n^+. This process is described in detail for the more general case of Bessis groups with complemented presentations in section 3.1, however in this case it can be practically summarised by applying the complete rewriting system:

$$\begin{aligned}
&\sigma_i \sigma_j^{-1} \to \sigma_j^{-1} \sigma_i^{-1} \sigma_j \sigma_i, \ \text{when} \ \ |i-j| = 1 \\
&\sigma_i \sigma_j^{-1} \to \sigma_j^{-1} \sigma_i, \ \text{when} \ \ |i-j| \geq 2 \\
&\sigma_i^{-1} \sigma_i \to \emptyset
\end{aligned}$$

We know that there exists a minimum $k \in \mathbb{N}$ such that $\Delta^k \succ v$, that is $uv = \Delta^k$ for some $u \in B_n^+$, so that $x = \Delta^{-k} uw$. Now $uw \in B_n^+$ so we may proceed as above to again obtain a normal form.

A pictorial example of this process can be seen in Figure 1.6.

Since this form is uniquely determined by the braid (rather than the braid word), two braid words are equivalent if and only if they are identical when put into normal form.

Furthermore, since the normal form is a complete invariant of braids the integers p, and r are invariants of the braid, and from these we draw the following functions on braids which will play an important role in the conjugacy problem:

Definition 1.3.2 *Given a braid g in normal form, $\Delta^p x_1 \cdots x_r$, we define the* ***infimum****,*

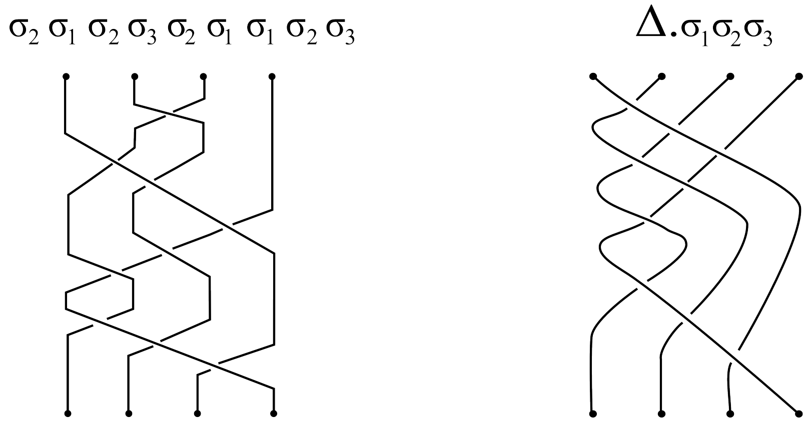

Figure 1.6: A braid put into normal form.

supremum *and* ***length*** *of g as follows:*

$$inf(X) = p$$
$$sup(X) = p + r$$
$$l(X) = r$$

1.4 The Conjugacy Problem

We begin by recalling the two distinct aspects of the conjugacy problem, detection and search. The conjugacy detection problem asks whether, given two elements of a group, we can determine if they are conjugate; while the conjugacy search problem asks us, given two conjugate elements, to give a conjugating element. Our primary focus is on detection, however the existing solutions all solve the search problem as a result of the detection solution. We begin with Garside's solution, and go on to mention a couple of improved versions.

1.4.1 Garside's Solution

Garside produced the first solution to the conjugacy problem in the same paper in which he gave the solution to the word problem described in the previous section ([23]). His method consisted of describing a subset of the conjugacy class of a braid which is finite, can be effectively computed, and is a complete invariant of conjugacy class. That set is defined as follows:

Definition 1.4.1 (Summit Set) *Let the conjugacy class of the element $g \in B_n$ be denoted $Cl(g)$. The* ***summit set*** *of a braid g, denoted $SS(g)$ is the set of all conjugate braids, which when put into normal form have the maximal infimum, that is:*

$$SS(g) = \{h \in Cl(g) \mid inf(h) \geq inf(x) \text{ for all } x \in Cl(g)\}$$

The following lemma will allow us to compute the summit set of a given braid.

Lemma 1.4.2 *Let $x = \Delta^p x_1 \cdots x_r$ and $y = \Delta^q y_1 \cdots y_s = v^{-1}xv$, with v a positive word $v = v_1 \cdots v_t$ in normal form, with the initial v_i possibly equal to Δ. Then $q \geq p$.*

Proof. The result follows trivially from well known properties of Δ if $v_1 = \Delta$.
So we assume $v_1 \neq \Delta$ and choose a simple braid u such that $v_1 u = \Delta$, then:

$$v_1^{-1} x v_1 = v_1^{-1} v_1 u \Delta^{p-1} x_1 \cdots x_r v_1 = \Delta^{p-1} \tilde{u} x_1 \cdots x_r v_1$$

where $\tilde{u}$ is u if p is odd, u with each σ_i replaced by σ_{n-i} if p is even, since it's known that $\sigma_i \Delta = \Delta \sigma_{n-i}$. This yields the following:

$$(v_2 \cdots v_t)^{-1} \Delta^{p-1} \tilde{u} x_1 \cdots x_r v_1 v_2 \cdots v_t = y = \Delta^q y_1 \cdots y_s$$

since all of the v_i are positive, and $q \geq p$, this implies that $\tilde{u} x_1 \cdots x_r v_1 v_2 \cdots v_t$ contains Δ. Furthermore Δ cannot be in $v_2 \cdots v_t$ since then the normal form of v would begin with $v_1 = \Delta$, so $\tilde{u} x_1 \cdots x_r v_1$ contains Δ.
So $v_1^{-1} x v_1 = \Delta^{p-1} \tilde{u} x_1 \cdots x_r v_1$ must have infimum at least p. ■

Given a braid g in normal form, with $inf(g) = p$, we can now compute its summit set as follows. Conjugate g by each simple braid, and record those distinct from g with infimum greater than or equal to p as $g_1, \ldots, g_s$. Repeat this process with each g_i, and record the resulting words as $g_{s+1}, \ldots$. Take the subset of $\{g_i\}$ consisting of those words with maximal infimum, this gives the summit set. In order to see that conjugating only by a sequence of simple braids gives all possible conjugates note that if we conjugate a word g by a word $h = \Delta^m \cdot x_1 \cdots x_r$ we have:

$$h^{-1} g h = x_r^{-1} (x_{r-1}^{-1} \cdots (x_1^{-1} (\Delta^{-1} \cdots (\Delta^{-1} g \Delta) \cdots \Delta) x_1) \cdots x_{r-1}) x_r$$

which is a sequence of conjugations by simple braids.

By lemma 1.4.2 all braids in the summit set can be reached by a sequence of conjugations by simple braids such that at each stage the infimum does not decrease.

We define the **index length**, $\mathrm{il}(X)$, of a braid word X in the Artin generators to be the sum of the indices of the those generators.

We next show that the summit set is finite:

Lemma 1.4.3 *Let p, t be integers. The set of all braids g with $inf(g) \geq p$ and with normal form having index length t is finite.*

Proof. Suppose the braid h, satisfying the conditions of the theorem, is written in normal form as $h = \Delta^m x_1 \cdots x_r$; and that $x_1 \cdots x_r = y_1 \cdots y_s$ where each $y_i = \sigma_j$ for some $j = 1, \ldots, n-1$.

Put $d = \mathrm{il}(\Delta)$. Then $t = md + s$, so:

$$m = \frac{t-s}{d} \leq \frac{t}{d}.$$

Thus the number of possible infimums is finite, and for each fixed power of Δ we have $r = t - md$, which yields a finite number of possibilities. ■

Corollary 1.4.4 *The summit set of any braid word is finite.*

Proof. The set of braids $g_1, \ldots$ is finite by Lemma 1.4.3, since conjugating does not alter index length. The summit set of a braid is therefore a subset of a finite set. ■

It remains to show that the summit set determines the conjugacy class of a braid.

Theorem 1.4.5 *Two braids g, and h are conjugate if and only if their summit sets are equal, that is: $SS(g) = SS(h)$.*

Proof. Suppose $SS(g) = SS(h)$, then choose some element $x \in SS(g) = SS(h)$. Obviously g and h are both conjugate to x, and so g is conjugate to h.
Now we prove the converse, suppose g and h are conjugate, and choose elements of their summit sets:

$$x = \Delta^p x_1 \cdots x_r \in SS(g)$$
$$y = \Delta^q y_1 \cdots y_s \in SS(h).$$

We may assume that $q \geq p$.
Obviously x and y are conjugate, suppose the conjugating element is $v = v_1 \cdots v_r$ in normal form, so that $v^{-1}xv = y$.
Then by Lemma 1.4.2, $v_1^{-1}xv_1 \in SS(g)$, and similarly by repeated applications of the same lemma,

$$v_k^{-1} \cdots v_1^{-1} x v_1 \cdots v_k \in SS(g)$$

for each k.
Thus for any $z \in SS(h)$, $z \in SS(g)$ and the proof is complete. ■

Thus given two braids g, and h, we can determine whether they are conjugate simply by calculating and comparing their summit sets. In particular, if two braids g and h are conjugate and we have calculated the summit set of each, we will have for each $x \in SS(g)$, a sequence of simple braids $v_1, \ldots v_m$ such that $v_m^{-1} \cdots v_1^{-1} g v_1 \cdots v_m = x$; and for each $y \in SS(h)$, a sequence simple braids $w_1, \ldots, w_n$, such that $w_n^{-1} \cdots w_1^{-1} B w_1 \cdots w_n = y$. Since g and h are conjugate, we may assume $x = y$, and so $g = v_1 \cdots v_m w_n^{-1} \cdots w_1^{-1} y w_1 \cdots w_n v_m^{-1} \cdots v_1^{-1}$, that is, we have not only conjugacy, but a conjugating element. So the summit set method solves also the conjugacy search problem.

The speed of this algorithm is bounded by the size of the summit set which is exponential with respect to the length of the word.

1.4.2 The Super Summit Set and Ultra Summit Set

There have been two improvements made to Garside's solution, which use the same method of finding a finite computable set which determines conjugacy class, but use refinements of the summit set to speed up the algorithm.

The first of these came in [22] in the form of the super summit set (SSS), which is defined as follows:

Definition 1.4.6 (Super Summit Set) *The super summit set of a braid g, denoted $SSS(g)$, is the set of all conjugate braids, which when put given in normal form have the maximal infimum and minimal length, that is:*

$$SSS(g) = \{h \in Cl(g) \mid inf(h) \geq inf(x) \text{ and } l(h) \leq l(x) \text{ for all } x \in Cl(g)\}$$

Corollary 4.2 in [22], demonstrates that given an element of the super summit set of a braid, the entire set can be computed by conjugating only with simple braids (much as in Lemma 1.4.2).

In order to compute the super summit set of a braid we need then only find one element thereof, which can be achieved through a process called cycling.

Definition 1.4.7 *Let $x = \Delta^p x_1 \cdots x_r$ be a braid word in normal form, then we define its* ***initial factor*** *to be $\Delta^p x_1 \Delta^{-p}$, and its* ***cycling****, $c(x)$, to be the conjugate of x by this initial factor, that is $c(x) = \Delta^p x_2 \cdots x_r \Delta^p x_1 \Delta^{-p}$. Similarly we can define the* ***final factor*** *of x to be x_r, and the* ***decycling****, $d(x)$, to be the conjugate of x by x_r^{-1}, that is $d(x) = x_r \Delta^p x_1 \cdots x_{r-1}$.*

Intuitively then, given a braid word in normal form cycling shifts the initial factor to the end, and decycling shifts the final factor to the beginning.

Lemma 4.3 and Corollary 4.4 in [22] tell us the following. Suppose for a given n, the index length of $\Delta \in B_n$ is m, then for a braid g with $l(g) = r$, a sequence of at most rm cyclings and decyclings of g yields an element of $SSS(g)$.

So the super summit set determines conjugacy, is computable and is a subset of the summit set. We thus have a solution to the conjugacy problem that is strictly faster than using the summit set.

The best solution known is that given by Gebhardt in [24], which further restricts from the SSS to the ultra summit set (USS), making use of cyclings to do so:

Definition 1.4.8 (Ultra Summit Set) *The ultra summit set of a braid g is given by all* *those elements of* $SSS(g)$ *which form a closed orbit under cycling, that is:*

$$USS(g) = \{h \in SSS(g) \mid c^k(h) = h \text{ for some } k \in \mathbb{N}\}$$

An element of the ultra summit set of a braid g can be found by taking any element of the super summit set, and cycling until we find a repetition. Once we have found such an element, we can once again find all others by conjugating only by simple braids (see [24] Theorems 1.17 and 1.21). So we have once again a set of braids that is computable, finite as a subset of the super summit set, and determines conjugacy.

Examples exist where the USS of a braid is strictly smaller than its SSS which is in turn strictly smaller than its summit set, so these reductions give a demonstrable improvement to the original solution. However there remains no polynomial bound on the size of even the USS, and so none of these attempts gives a polynomial solution.

1.5 Progress towards a Polynomial Solution

In this, the final section of this chapter, we discuss more recent work on the conjugacy problem in braid groups, mainly focussing on a sequence of papers by Birman, Gebhardt, and Gonzales-Meneses, [10][11][12], in which they present a scheme for solving the conjugacy problem in polynomial time. While this scheme has yet to yield such a solution it does reduce the problem to a number of possibly simpler questions.

1.5.1 The Braid Group as a Mapping Class Group

In this subsection, we begin by considering another alternative description of the braid group, and explaining its equivalence. The mapping class group of a surface is defined as follows:

Definition 1.5.1 (Mapping Class Group) *The* ***mapping class group*** *of a surface* S*,* $MCG(S)$*, is the group of isotopy-classes of automorphisms of* S *with multiplication by composition. That is:*

$$MCG(S) = \frac{Aut(S)}{Aut_0(s)}$$

where $Aut(S)$ *is the group of automorphisms on* S*, and* $Aut_0(S)$ *is the path component containing the identity of* $Aut(S)$*.*

We now establish some notation, we call D the 2-dimensional disk in $\mathbb{R}^2$, and D_n the disk with n-punctures, that is with n points removed.

Theorem 1.5.2 *The braid group on* n*-strands,* B_n *is isomorphic to* $MCG(D_n)$*, the mapping class group of the* n*-punctured disk.*

This isomorphism can be envisioned as follows, if we imagine a braid positioned vertically, and force the punctured disk D_n down over it, it induces an automorphism of the punctured disk.

1.5.2 The Thurston-Nielsen Classification

The fact that the braid group can be considered as a mapping class group, allows us to apply to it the Thurston-Nielsen classification of mapping classes, which in the context of the braid group can be given as follows:

Theorem 1.5.3 *[10] Let $g \in B_n$; then after some isotopy g belongs to precisely one of the following disjoint classes:*

1. *Periodic, if some power of g is a power of Δ^2.*
2. *Pseudo-Anosov, if neither g nor any power of g fixes the isotopy class of any simple closed curve in D_n.*
3. *Reducible, if there exists a family, $\{\alpha_a\}_{a\in A}$, of simple closed curves in D_n preserved by g (up to isotopy). Such a family is called a **reduction scheme**. A reduction scheme is **complete** if when we split open D_n along its curves, the restriction of g to the resulting disks is in each case either periodic or pseudo-Anosov.*

Furthermore it is clear that conjugation preserves this classification, so we may divide the conjugacy problem into three cases, namely those of periodic, reducible and pseudo-Anosov braids. However in order for this to lead to an effective algorithm, we need to be able to find the type of a mapping class in polynomial time.

Methods for determining whether a mapping class is periodic are very simple and are given in [25], we will give more specifics in the following subsection (1.5.4).

Finding whether non-periodic braids are reducible or pseudo-Anosov is more difficult. The first method for determining reducibility was that of [8], using the methods of train tracks and zipping.

There have been more recent solutions using the machinery of summit sets to reduce the number of possibly simply closed curves that need to be checked to what are called standard circles, and in [15], it was shown that reducibility can be determined in polynomial time, although the exact bound is yet to be found.

1.5.3 Reducible Braids

Suppose we have some reducible braid, g, with a complete reduction scheme $\{\alpha_a\}_{a\in A}$. Consider a representative of the isotopy class of one of the simple closed curves α_a which is preserved by the braid. Considering g as a geometric braid, over the course of g, this curve traces out a cylinder or tube, and this tube contains $k_a \geq 1$ strands of g. So we may associate to α_a a braid in B_{k_a}, which we call $int_a(g)$, the interior braid of α_a. Repeating this with appropriate choices of representative curves for each $a \in A$, gives several tubes each containing one or more of the original strands forming an interior braid for that tube, and the tubes themselves are braided to form what is called a tubular braid on $|A|$ strands, denoted $tub(g)$. So the reducible braid can be broken down into one tubular braid which must be

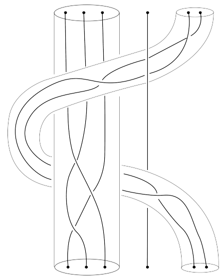

Figure 1.7: A reducible braid split into a tubular braid and component interior braids.

periodic or pseudo-Anosov (as otherwise the reduction scheme would be incomplete), and for each tube an interior braid, see Figure 1.7 for an example.

If two reducible braids g and h are conjugate then $tub(g)$ and $tub(h)$ must be conjugate, so when considering conjugacy we assume some conjugation has been applied so that the tubular braids are actually equal, then we have the following result from [25]:

Proposition 1.5.4 *Let g and h be two reducible braids with complete reduction schemes $\{\alpha_a\}_{a\in A}$ and $\{\beta_b\}_{b\in B}$ respectively, such that $tub(g) = tub(h)$. Then g and h are conjugate if and only if each $int_a(g)$ is conjugate to some $int_b(h)$ and there is some element of the centraliser of $tub(g)$ which takes the tube containing each $int_a(g)$ to the tube containing the corresponding $int_b(h)$ in h*

So the conjugacy problem can be broken into the tubular braids, which are guaranteed to be non-reducible, and to the interior braids which can themselves be reduced until we are only dealing with periodic and pseudo-Anosov braids.

Before we move onto these other cases, we note that in order to enact this restriction of the problem we need to be able to find complete reduction schemes of reducible braids. Unfortunately while it is known that a polynomial algorithm for determining whether a braid has a non-empty reduction scheme exists, there is no known fast way of finding the scheme itself. This results in the first open problem of this strategy.

Problem 1.5.5 *Given a reducible braid, find a reduction scheme in polynomial time.*

1.5.4 Periodic Braids

Periodic braids are the simplest of the three cases, and a polynomial algorithm for determining conjugacy among periodic braids in B_n can be found in [12]

Consider the braids $\delta = \sigma_1\sigma_2\cdots\sigma_{n-1}$ and $\gamma = \sigma_1^2\sigma_2\cdots\sigma_{n-1}$ (in the braid group on n strands). Simple calculation shows that $\delta^n = \gamma^{n-1} = \Delta^2$, so that both braids are periodic. The following result (Theorem 4 in [12]) uses these braids to characterise all periodic braids and is the key to solving the conjugacy problem for periodic braids.

Theorem 1.5.6 *Every periodic braid is conjugate to either some power of δ, or some power of γ.*

We can determine which conjugacy class a periodic braid X belongs to in the following manner. If g is conjugate to γ^k, then $g^{n-1} = \Delta^{2k}$, and if g is conjugate to δ^k, then $g^n = \Delta^{2k}$.

This suffices to solve the conjugacy problem among periodic braids.

1.5.5 Pseudo-Anosov Braids

Assuming a solution to Problem 1.5.5 is found, the conjugacy problem can then be reduced to the pseudo-Anosov case. The strategy expounded in [10] and [11] focuses on trying to find a upper bound to the size of the USS when the braid in question is pseudo-Anosov.

Definition 1.5.7 *We call a braid X with normal form $\Delta^p x_1 \cdots x_r$* ***rigid*** *if the word $x_r \cdot \Delta^p x_1 \Delta^{-p}$ is in normal form.*

Proposition 1.5.8 *[10] Let $g \in B_n$ be a pseudo-Anosov braid, then for some $k \in \mathbb{N}$, g^k is rigid. Furthermore such a k can be found in polynomial time.*

Clearly if a braid is rigid, then any power of that braid will also be rigid, since the initial and final factors involved are unchanged.

Proposition 1.5.9 *[25] Given a braid $g \in B_n$, if there exist two braids $x, y \in B_n$ such that $x^k = y^k = g$, then x is conjugate to y*

The previous proposition may be restated as: braids have unique roots up to conjugacy. Taking the two preceding propositions we have the following: given any two pseudo-Anosov braids g and h we can find n, m such that g^n and h^m are rigid. Thus g and h are conjugate if and only if g^{nm} and h^{nm} are conjugate. So we may restrict further from the case of pseudo-Anosov braids, to that of rigid pseudo-Anosov braids.

This brings us to the second and final gap in the proposed solution.

Problem 1.5.10 *Give an algorithm determining conjugacy among rigid pseudo-Anosov braids in polynomial time.*

Chapter 2

The faithful representation, and conjugacy in braid groups

In the first section of this chapter we define the faithful representation of the braid group, also known as the Lawrence-Krammer-Bigelow, or LKB representation, and mention some of its properties. In the second we see how it can be applied to the conjugacy problem in the braid group. We then show that for pseudo-Anosov rigid braids satisfying a certain condition on their eigenvalues the conjugacy problem is solvable in polynomial time.

2.1 The LKB representation

The action of this representation comes from the action of the braid group on the second homology group of the universal cover of the configuration space of n points in the complex plain.

The representation was first introduced in Lawrence's paper [28], before it was shown to be faithful independently by Bigelow ([9]) and Krammer ([27]). We will be using throughout the notation and methodology of [27], to expound the relationship between the representation and the conjugacy problem.

The representation gives an action of the braid group of n-strands on a vector space of dimension $m = \frac{n(n-1)}{2}$ over the ring $K = \mathbb{Q}[q^{\pm 1}, t^{\pm 1}]$. The proof of faithfulness given by Krammer holds if q is any real number in $(0,1)$, so we fix q and think of K as $\mathbb{R}[t^{\pm 1}]$. We may write elements of this ring as $\lambda = \sum_{i=k}^{l} a_i t^i$, for some $k, l \in \mathbb{Z}$ and $a_i \in \mathbb{R}$ .The valuation of such λ, $val(\lambda)$, is then the least i such that $a_i \neq 0$ (by convention $val(0) = \infty$). Furthermore we have a ordering, $\prec$, on K given by $0 \prec \lambda$ if $a_{val(\lambda)} > 0$ (and $0 \preceq \lambda$ if $a_{val(\lambda)} > 0$ and or $\lambda = 0$). This yields a topology on K whose open sets are intervals $(a,b) = \{c \in K \mid a \prec c \prec b\}$.

In [27], Krammer gives matrices for the representation when $K^{\frac{n(n-1)}{2}}$ has the following basis:

$$\{x_s \mid s \in Ref\}, \text{where}$$
$$Ref = \{(i,j) \in \Sigma_n \mid i < j\}$$

namely $\rho : B_n \to Gl(m, K)$ is given by:

$$\begin{aligned}
\rho(\sigma_k)x_{(k,k+1)} &= tq^2 x_{(k,k+1)}; && \\
\rho(\sigma_k)x_{(i,k)} &= (1-q)x_{(i,k)} + qx_{(i,k+1)}, && i < k; \\
\rho(\sigma_k)x_{(i,k+1)} &= x_{(i,k)} + tq^{k-i+1}(q-1)x_{(k,k+1)}, && i < k; \\
\rho(\sigma_k)x_{(k,j)} &= tq(q-1)x_{(k,k+1)} + qx_{(k+1,j)}, && k+1 < j; \\
\rho(\sigma_k)x_{(k+1,j)} &= x_{(k,j)} + (1-q)x_{(k+1,j)}, && k+1 < j; \\
\rho(\sigma_k)x_{(i,j)} &= x_{(i,j)} && i < j < k \text{ or } k+1 < i < j; \\
\rho(\sigma_k)x_{(i,j)} &= x_{(i,j)} + tq^{k-i}(q-1)^2 x_{(k,k+1)}, && i < k < k+1 < j.
\end{aligned}$$

So we write the action of B_n on the left, and for $v \in K^m$ and $g \in B_n$, we will write $g(v)$ for $\rho(g)(v)$.

2.2 Relation between representation and conjugacy

In this section we will define some key aspects which were intrumental in Krammer's proof, and will crop up repeatedly in the following.

The main object we shall be dealing with is a convex cone within the faithful representation upon which the braid group acts. To describe this convex cone, we begin by defining the following subset of the representation which we shall term the fundamental chamber of the convex cone:

Definition 2.2.1 (The fundamental chamber) *We recall that the representation is defined over the ring $\mathbb{R}[t^{\pm}]$. The* ***fundamental chamber*** *C is defined as follows:*

$$C = \{\Sigma_{s \in Ref} a_s x_s \mid a_s \in \mathbb{R}_{>0} + t\mathbb{R}[t]\}$$

where Ref is the set by which the basis of the representation is labelled.

We define the convex cone, U, to be the orbit of the fundamental chamber under the action of the braid group, that is:

$$U := \bigcup_{g \in B_n} g \cdot C$$

The fact that this set is in fact a convex cone, and the following theorem are proven in [27].

Theorem 2.2.2 *Any union of chambers in the cone U is convex if and only if the corresponding subgraph of braids in the Cayley graph of the braid group, with generators the simple braids, (that is, the braids which map the fundamental chamber to the chambers in question) is convex in the graph sense.*

Another way to think of this is that any two chambers in the subset may be joined by a sequence of simple braids which together give a normal form.

Given this property we may advance a scheme for using the LKB representation to solve the conjugacy problem for pseudo-Anosov braids. Recall that we may restrict to rigid pseudo-Anosov braids. Let $g = x_1 \dots x_r$ be a rigid pseudo-Anosov braid written in normal form. Then the set $Z_l := \bigcup_{i=-l,\dots,l} g^i C$, is convex for any finite integer l, since by rigidity g^k is already in normal form.

We choose some point in $z \in C$. Then clearly $g^i(z) \in g^i C$ for $i \in \mathbb{Z}$. Let $L_l := \{g^{-l}(z) + t(g^l(z) - g^{-l}(z)) \mid t \in [0,1] \subset \mathbb{R}\}$, that is L_l is the line segment in U connecting $g^{-l}(z)$ and $g^l(z)$, for $l \in \mathbb{Z}$. As a result of theorem 2.2.2 we have that $L_l \cap g^i C \neq \emptyset$ for $i = -k, -k+1, \dots, k-1, k$.

Consider the infinite extension of Z_l. $Z_\infty := \bigcup_{i \in \mathbb{Z}} g^i C$. If there were a straight line segment in U between eigenvectors of g passing through all of the chambers in this set and contained in their union, that is:

(2.1) There exists a line segment $L \subset U$ between eigenvectors of g such that $L \subset Z_\infty$ and $L \cap g^i C \neq \emptyset$ for all $i \in \mathbb{Z}$.

then it would determine an infinite normal form in the braid group. That is an infinite string of simple braids $\dots, x_{r-1}, x_r, x_1, \dots, x_{r-1}, x_r, x_1, \dots$, such that each pair $x_i x_{i+1}$ is in normal form. Now suppose $h \in USS(g)$, then h has the same eigenvalues and eigenvectors as g, and the line L is again a line between eigenvectors of h such that $L \subset Z_\infty$ and $L \cap g^i C \neq \emptyset$ for all $i \in \mathbb{Z}$. From this we may conclude that g and h share the same infinite normal form, and therefore that g and h belong to the same orbit under cycling in $USS(g)$, which consequently must consist of a single orbit. So if such a line exists for g then $USS(g)$ can be found in polynomial time.

In the next section we will give conditions on the eigenvalues of a braid under the LKB representation, which if satisfied necessitate the existence of such a line. In particular the line will exist as a kind of limit of the lines L_l.

2.3 Eigenvalue related methods

We begin by defining the notion of exceptional extremal (maximum and minimum) eigenvalues of a braid.

Definition 2.3.1 *We say that a simple eigenvalue, λ, of a braid g (under the LKB representation) is an* ***exceptional maximum eigenvalue*** *if the following set:*

(2.2) $\left\{\left|\frac{\mu}{\lambda}\right|^n \mid n \in \mathbb{N}\right\}$

*is unbounded for each eigenvalue or generalised eigenvalue μ of g, which is different from λ. Similary a simple eigenvalue ν is an **exceptional minimum eigenvalue** if the set:*

(2.3) $\left\{\left|\frac{\nu}{\mu}\right|^n \mid n \in \mathbb{N}\right\}$

is unbounded for each eigenvalue or generalised eigenvalue of g, μ, which is different from ν.

Since the representation is defined over $K = \mathbb{R}[t^{\pm 1}]$, by proposition 29 of [16], the eigenvalues are in the algebraic closure of K, that is the field of Puiseux series in t over $\mathbb{R}$, which we denote $\bar{K}$, and may be written in the form $\lambda = \sum_{i=k}^{+\infty} a_i t^{i/j}$, for some $j \in \mathbb{Z}$ dependent on λ. The valuation of such λ, $val(\lambda)$, is then the least i/j such that $a_i \neq 0$ (by convention $val(0) = \infty$). Furthermore we have a ordering, $\prec$, on $\bar{K}$ given by $0 \prec \lambda$ if $a_{val(\lambda)} > 0$ (and $0 \preceq \lambda$ if $a_{val(\lambda)} > 0$ and or $\lambda = 0$). This yields a topology on $\bar{K}$ whose open sets are intervals $(a, b) = \{c \in \bar{K} \mid a \prec c \prec b\}$. The conditions of definition 2.3.1 can be understood as saying that there are minimum and maximum simple eigenvalues (with respect to $\preceq$) which have a unique valuation among all simple and generalised eigenvalues.

In this section we will show that pseudo-Anosov rigid braids which have both maximum and minimum exceptional eigenvalues satisfy (2.1), and in particular the line segments L_l converge to L_∞ in a meaningful sense.

The first step towards this result is showing that the fundamental chamber, C is closed. We write $(a_1, \ldots, a_m)$ for the element $\sum_{s \in Ref} a_s x_s$ of K^m.

Proposition 2.3.2 *Recall the fundamental chamber $C = \{\Sigma_{s \in Ref} a_s x_s \mid a_s \in \mathbb{R}_{>0} + t\mathbb{R}[t]\}$. C is closed in $(K)^m$.*

Proof. Suppose $v = (v_1, \ldots, v_m)$ is a limit point of C, so that any open neighbourhood of v contains an element of C; and that v is not in C itself.
Since v is not in C, we have two possibilities: either all coordinates of v have valuation 0, but some coordinate (we may assume it is the first) of v is negative, that is $val(v_i) = 0$ for $i = 1, \ldots, m$ and $v_1 \preceq 0$; or some coordinate of v has valuation non-zero, we assume it is the first and that its valuation is k, that is $val(v_1) = k \neq 0$. In the first case we consider the following open neighbourhood of v:

$$W_v := \{(w_1, \ldots, w_m) \mid val(w_1) = 0, w_1 \preceq 0\}$$

Since the first coordinate of any point in W_v is negative, we have $W_v \cap C = \emptyset$. In the second case we look at:

$$W'_v := \{(w_1, \ldots, w_m) \mid val(w_1) = k\}$$

Now the first coordinate of any point in W'_v has valuation $k \neq 0$ so again $W'_v \cap C = \emptyset$. Thus a point can only be a limit point of C if it is in C itself, and so C is closed. ■

Next under the assumption that our braid g admits exceptional extreme eigenvalues we show that the associated exceptional eigenvectors are inside the closure of the convex cone U.

Proposition 2.3.3 *Let $g \in B_n$ be a braid admitting exceptional maximum eigenvalue, λ_1 and exceptional minimum eigenvalue λ_m, with associated eigenvectors e_1 and e_m respectively. There exist points $x, y \in U$ such that the sequences $g^k(x)$ and $g^{-k}(y)$ in U converge to e_1 and e_m respectively. In particular e_1 and e_m are in the closure $\bar{U}$ of the convex cone $U = \bigcup_{g \in B_n} gC$.*

Proof. Suppose that the eigenvalues of g are $\lambda_1, \lambda_2, \ldots, \lambda_m$, and let $e_1, e_2, \ldots, e_m$ be a basis of eigenvectors and generalized eigenvectors associated to those λ_i.
Choose some point $x = \sum_{i=1}^{m} a_i e_i \in U$ such that $a_1 \neq 0$. Such a point exists since U is n-dimensional.
Then the action of g can be described as follows:

(2.4) $g^k(x) = \sum_{i=1}^{m} \lambda_i^k a_i e_i p_i(k)$

where λ_i is the eigenvalue of e_i and p_i is some polynomial. Furthermore, since λ_1 and λ_m are simple eigenvalues, we have $p_1 = p_m = 1$. The action of g on U naturally gives an action on U/K^*.
We establish the following notation, the equivalence class of $x = \sum_{i=1}^{n} a_i e_i$ in U/K^* is denoted by:

$$[x] = [a_1, \ldots, a_n]$$

Then (2.4) can be restated as follows:

$$\begin{aligned} g^k([x]) &= \left[\lambda_1^k a_1, \lambda_2^k a_2 p_2(k), \ldots, \lambda_{m-1}^k a_{m-1} p_{m-1}(k), \lambda_m^k a_m\right] \\ &= \left[1, \frac{\lambda_2^k a_2 p_2(k)}{\lambda_1^k a_1}, \ldots, \frac{\lambda_{m-1}^k a_{m-1} p_{m-1}(k)}{\lambda_1^k a_1}, \frac{\lambda_m^k a_m}{\lambda_1^k a_1}\right] \\ &= \left[1, \left(\frac{\lambda_2}{\lambda_1}\right)^k \frac{a_2 p_2(k)}{a_1}, \ldots, \left(\frac{\lambda_{m-1}}{\lambda_1}\right)^k \frac{a_{m-1} p_{m-1}(k)}{a_1}, \left(\frac{\lambda_m}{\lambda_1}\right)^k \frac{a_m}{a_1}\right] \end{aligned}$$

Suppose W_{e_1} is some open neighbourhood of $[e_1] = [1, 0, \ldots, 0]$, then W_{e_1} contains some subset of the form:

$$\{[x_1, \ldots, x_m] \mid x_1 \in (1 - \epsilon_0, 1 + \epsilon_0) \text{ and } x_i \in (-\epsilon_i, +\epsilon_i) \text{ for } i \neq 1\}$$

Then for some sufficiently large k, $g^k([x])$ is in W_{e_1}, since each $\left(\frac{\lambda_1}{\lambda_i}\right)^k$ tends to 0, by (2.2). Therefore any open neighbourhood of $[e_1]$ contains a point of U/K^*.

Thus $[e_1]$ is a limit point of U/K^* and so $[e_1]$ is in $\overline{U/K^*}$. Therefore $[e_1]$ is in $\bar{U}/K^*$.

Finally we see that $e_1 \in \bar{U}$, as required.

The proof that $e_m \in \bar{U}$, is similar, replacing g^k with g^{-k}, and (2.2) with (2.3). ■

So we have two eigenvectors of g, e_1 and e_m, sitting on the boundary of U, and we consider the straight line segment between these two vectors: $L_\infty = \{e_1 + t(e_m - e_1) \mid t \in [0,1] \subset \mathbb{R}\}$. We show that this line segment meets the conditions of L in (2.1).

To this end we move from considering the fundamental chamber C and straight line segments to the cone closure of the fundamental chamber, and subspaces of K^m of dimension 2. We begin with definitions and some preliminary propositions.

Definition 2.3.4 *Let D be some convex set in K^m, we define the cone closure of D, D^{cone} to be the intersection of all convex cones containing D.*

In particular we are interested in the cone closure of chambers in U.

Proposition 2.3.5 *Let C be the fundamental chamber in the convex cone U. Then $C^{cone} = \bigcup_{i\in\mathbb{Z}} \Delta^i C$.*

Proof. Note that since each $\Delta^i C$ is convex, and the subgroup $\langle\Delta\rangle \subset B_n$ forms a convex subset of the Cayley graph of B_n, we have convexity for the whole subset $\bigcup_{i\in\mathbb{Z}} \Delta^i C$.
Thus it only remains to show that this set is closed under scalar multiplication in order to see that it is a convex cone.
Let $v \in \bigcup_{i\in\mathbb{Z}} \Delta^i C$ and $a \in K$; then if $a \in \mathbb{R}$ then v remains in the same $\Delta^i C$, and if a has valuation $k > 0$ then the valuation of each coefficient of v is increased by k, and since the action of Δ is just multiplication by t up to powers of q, this just shifts us to another $\Delta^i C$, and we remain in the overall set.
Therefore the set under consideration is a convex cone, and since any convex cone containing C must necessarily contain this set, it must be C^{cone}. ■

We can extend our understanding of the cone closures of convex chambers to the chamber associated to a general element $g \in B_n$.

Proposition 2.3.6 *Let gC be the convex chamber associated to the element $g \in B_n$. The the cone closure of gC is:*

$$(gC)^{cone} = \bigcup_{i\in\mathbb{Z}} (g\Delta^i g^{-1})gC$$

Proof. As before we need only show closure under scalar multiplication.
Suppose $v \in gC$, then:

$$\begin{aligned} g^{-1}v &\in C \\ a \cdot g^{-1}(v) = g^{-1}(a \cdot v) &\in \Delta^k C \text{ for some } k \in \mathbb{Z} \text{ by Proposition 2.3.5} \\ a \cdot v &\in g\Delta^k C = (g\Delta^k g^{-1})gC, \end{aligned}$$

therefore the space in question is closed under scalar multiplication and the proof is complete. ■

Definition 2.3.7 *We define $Gr(2, K^m)$ to be the space of all 2-dimensional subspaces of K^m, and*

$$Q := \{V \in Gr(2, K^m) \mid V \cap C \neq \emptyset\}$$

to be the set of all 2-dimensional subspaces intersecting the fundamental chamber C.

We describe points in $Gr(2, K^m)$, by pairs of spanning vectors, for example $W = \langle w_1, w_2 \rangle \in Gr(2, K^m)$ where w_1 and w_2 are linearly independent vectors in K^m.

The basis of a topology on $Gr(2, K^m)$ is given by pairs of products of open intervals, that is $\langle \prod_{i=1}^m (a_i, b_i), \prod_{i=1}^m (c_i, d_i) \rangle$, where $a_i \preceq b_i$ and $c_i \preceq d_i$ and all are in K.

We define a sequence of subspaces in $Gr(2, K^m)$, analogous to the sequence line segments L_l. Let z be some point in C. Then we call P_l the subspace spanned by $g^l(z)$ and $g^{-l}(z)$, that is $P_l := \langle g_{-l}(z), g_l(z) \rangle$. Furthermore we call P_∞ the subspace spanned by e_1 and e_m, that is $P_\infty := \langle e_1, e_m \rangle$

We proceed as follows. We show that the set Q is closed in $Gr(2, K^m)$, that $P_l \in Q$ for all $l = 1, 2, \ldots$, and finally that this gives P_∞ as a limit point of Q. Finally we use this fact, along with the properties of convex cone closures in U to show that the line segment L_∞ satisfies (2.1).

Lemma 2.3.8 *The set $Q = \{V \in Gr(2, K^n) \mid V \cap C \neq \emptyset\}$ is closed in $Gr(2, K^m)$.*

Proof. Suppose $P = \langle p_1, p_2 \rangle$ is a limit point of Q, and extend p_1, p_2 to a basis $p_1, p_2, \ldots, p_m$ of K^m, and write an element $\sum_{i=1}^n a_i p_i$ as $(a_1, \ldots, a_n)$.
Let U_P be an open neighbourhood of P (specifically one given by products of open intervals centred on P, dealing solely with the basis of the topology for simplicity).

$$U_P = \langle p_1, p_2 \rangle + (0, 0, (-\epsilon_3, \epsilon_3), \ldots, (-\epsilon_n, \epsilon_n))$$

Since P is a limit point, there exists $P' \in U_P$ such that $P' \cap C \neq \emptyset$.
Choose some $x = (a_1, \ldots, a_n) \in P' \cap C$, and then consider $x' = (a_1, a_2, 0, \ldots, 0) \in P$.
Suppose $U_{x'}$ is some open neighbourhood of x'; having initially chosen U_P, sufficiently small we will have $x \in U_{x'}$. Thus x' is a limit point of C, and so $x' \in C$ by Proposition 2.3.2.
Since $x' \in P \cap C$, we can see that any limit point of Q is inside Q, and so Q is closed as required. ■

Lemma 2.3.9 *Let $P_l := \langle g^{-l}(z), g^l(z) \rangle$, for some $z \in C$ not an eigenvector of g. Then $P_l \cap C \neq \emptyset$, that is $P_l \in Q$.*

Proof. Clearly $L_l = \{g^{-l}(z) + t(g^l(z) - g^{-l}(z)) \mid t \in [0, 1] \subset \mathbb{R}\} \subset P_l$. As we mentioned above we have $L_l \cap C \neq \emptyset$ from theorem 2.2.2. This completes the proof. ■

Lemma 2.3.10 *The subspace of K^m generated by e_1 and e_m, P_∞, is a limit point of Q.*

Proof. We work with the basis of eigenvectors and generalised eigenvectors of g: $e_1, \ldots, e_m$, and write vectors as rows. Consider an open neighbourhood of $P_\infty = \langle e_1, e_m \rangle$:

$$U_W = \langle e_1 + ((-\epsilon_1, \epsilon_1), (-\epsilon_2, \epsilon_2), \ldots, (-\epsilon_m, \epsilon_m)), e_m + ((-\gamma_1, \gamma_1), (-\gamma_2, \gamma_2), \ldots, (-\gamma_m, \gamma_m)) \rangle$$

We may choose some $z = \sum_{i=1}^{m} a_i e_i \in C$ with a_1 and a_m not equal to zero. By proposition 2.3.3 the sequence $g^k(z)$ tends to e_1, and similarly that the sequence $g^{-k}(z)$ tends to e_m. So for sufficiently large k, the spanning vectors of P_k will be in any required open neighbourhoods of e_1 and e_m, so for sufficiently large k, $P_k \in U_W$.
So for any open neighbourhood of W, there exists an element of Q in that neighbourhood. ■

Corollary 2.3.11 $P_\infty = \langle e_\lambda, e_\mu \rangle \in Q$

Theorem 2.3.12 *Let $g \in B_n$ be a braid admitting exceptional extremal eigenvalues. Then the USS of g consists of a single orbit under cycling and conjugation by Δ.*

Proof. Since $P_\infty \in Q$, there exists some linear combination of e_1 and e_m which is in C. So the straight line between them passes through C^{cone}. Choosing some point x on this line and in C^{cone}, then $g^k(x) \in g^k C^{cone}$ is also on this line (it is still a linear combination of e_1 and e_m), for all k. The stretch of line between x and $g^k(x)$, that is, the convex hull $[x, g^k(x)]$ of x and $g^k(x)$ is entirely contained in $\bigcup_{i=0}^{k} (g^i C)^{cone}$. The complete line is equal to $\bigcup_{k\in\mathbb{Z}} [g^{-k}(x).g^k(x)]$, and is therefore contained in $\bigcup_{k\in\mathbb{Z}} (g^k C)^{cone}$ and has non-empty intersection with each $(g^k C)^{cone}$ for $k \in \mathbb{Z}$.
That is to say that g admits a line of the type described in (2.1), and by the argument advanced in section 2.2, the result holds. ■

Corollary 2.3.13 *For pseudo-Anosov rigid braids admitting exceptional extremal eigenvalues, the conjugacy problem is solvable in polynomial time.*

The following example shows that the set of rigid pseudo-Anosov braids admitting exceptional maximum and minimum eigenvalues, is non-empty, and so theorem 2.3.12 is a meaningful result.

Example 2.3.14 *The braid $\sigma_3\sigma_2\sigma_2\sigma_3\sigma_2\sigma_1\sigma_4\sigma_1\sigma_3\sigma_1\sigma_3\sigma_4\sigma_2\sigma_4\sigma_4\sigma_4\sigma_1 \in B_5$ is pseudo-Anosov, rigid, and admits exceptional maximum and minimum eigenvalues under the LKB representation.*

The converse of theorem 2.3.12, that is, that all braids with a unique orbit in their USS have exceptional maximum and minimum eigenvalues is disproved by the following counterexample:

Example 2.3.15 *The ultra summit set of $\sigma_1\sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_3\sigma_4\sigma_2\sigma_4\sigma_2\sigma_4\sigma_3\sigma_3\sigma_2\sigma_3\sigma_2 \in B_5$ consists of a single orbit under cycling and conjugation by Δ. The eigenvalues of this braid have valuations $6, 4, 2, 1, 0$ with multiplicities $1, 1, 2, 3, 3$, so there is no eigenvalue of unique minimum valuation.*

Chapter 3

Introduction to Bessis-Garside groups

In the first section of this chapter we introduce the concept of Bessis groups. We draw on the methods of Bessis in his papers [4]and [5], where he uses the well known Hurwitz action of the braid group on n strands B_n on the free group with n generators F_n, to construct Garside structures for Artin groups associated to finite Coxeter systems, and to the free group itself.

In the second section we describe all Bessis groups of rank 2, and show that they admit Garside structures. We also give representations for these groups over cyclic rings.

In the last section of this chapter we introduce cycle presentations, and give cycle presentations of the Bessis groups of rank 2.

3.1 Introduction

Let the free group F_n be given by the generators, $x_1, x_2 \ldots, x_n$. Then the Artin action of B_n on F_n, is defined by the action of the generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ of the braid group:

$$x_i.\sigma_j = \begin{cases} x_{i+1} & \text{if } i = j, \\ x_i^{-1} x_{i-1} x_i & \text{if } i = j+1, \\ x_i & \text{otherwise.} \end{cases}$$

the action of the inverses of the braid group generators is then:

$$x_i.\sigma_j^{-1} = \begin{cases} x_i x_{i+1} x_i^{-1} & \text{if } i = j, \\ x_{i-1} & \text{if } i = j+1, \\ x_i & \text{otherwise.} \end{cases}$$

as is often the case regarding the braid group in the literature the roles of the generators and their inverses (and consequently of their actions) are often reversed.

Let F_n^+ be the submonoid of F_n given by:

$$F_n^+ := \langle \{x_i.b \mid 1 \leq i \leq n, b \in B_n\} \rangle$$

Let $H < B_n$ be a subgroup of the braid group, then we associate to H a (normal) subgroup of F_n, N_H:

$$N_H := \left\langle x(x.g)^{-1} \mid x \in F_n, g \in H \right\rangle_{normal}$$

which we call simply N when there is no danger of ambiguity.

Definition 3.1.1 *Let G_H be defined as the quotient $G_H := F_n/N_H$, then G_H is a **Bessis group of rank** n; in particular we call G_H the Bessis group associated to H, which we may write as G for brevity.*

We call p the quotient map $p : F_n \to F_n/N_H$, and put $G^+ := pF_n^+$.

Definition 3.1.2 *Let G be a Bessis group. We put $\Delta := x_1x_2 \cdots x_nN \in G$, and define the ordering $\preceq$ on G by $x \preceq y$ if $x^{-1}y \in G^+$ for all $x, y \in G$. Then we call G a Bessis-Garside group if (G, Δ) is a Garside pair as defined in 1.3.1.*

Lemma 3.1.3 *A Bessis group G is a Bessis-Garside group whenever the ordering $\preceq$ given in definition 3.1.2 is a lattice ordering.*

Proof. Recall the definition of a Garside structure (1.3.1), and put $P := G^+$. To show that (G, Δ) is a Garside pair, we need to show that the ordering $\preceq$ is left-invariant, that $[1, \Delta]$ generates G, that $\Delta^{-1}P\Delta = P$, and that P is atomic. Suppose $x \preceq y$, then $x^{-1}y \in P$, so $x^{-1}z^{-1}zy \in P$ and therefore $zx \preceq zy$ as required. We have

$$\begin{aligned}
\Delta &= x_1x_2 \cdots x_nN \\
&= x_2(x_2^{-1}x_1x_2)x_3 \cdots x_nN \\
&= \ldots \\
&= x_n(x_n^{-1}x_{n-1}x_n)(x_n^{-1}x_{n-1}^{-1}x_{n-2}x_{n-1}x_n) \cdots (x_n^{-1}x_{n-1}^{-1} \cdots x_2^1x_1x_2 \cdots x_{n-2}x_{n-1}x_n)N
\end{aligned}$$

so that $x_1N, x_2N, \ldots x_nN \in [1, \Delta]$, and $[1, \Delta]$ generates G. The atoms of P are clearly $A = \{(x_1.g)N \mid g \in B_n\}$. To see that $P = \Delta^{-1}P\Delta$ we need only check that $\Delta^{-1}a\Delta \in P$ for all $a \in A$. Let $y = x_1.g$.

$$\begin{aligned}
\Delta &= yx_2.gx_3.g \\
&= x_2.gx_3.g((x_3^{-1}.g)(x_2.g)(x_3.g)), \text{so} \\
\Delta^{-1}a\Delta &= (x_3.g)^{-1}(x_2.g)^{-1}y^{-1}yx_2.gx_3.g((x_3^{-1}.g)(x_2.g)(x_3.g)) \\
&= (x_3^{-1}.g)(x_2.g)(x_3.g).
\end{aligned}$$

Finally let x be a word in P, then we may consider the index length of x as a word in x_i, and call this $il_{x_i}(x)$. It is a property of the Hurwitz action that $il_{x_i}(x_1.g) = 1$ for all $g \in B_n$, so any decomposition of x into atoms has at most $il_{x_i}(x)$ terms. ■

In the following we use the notions of complements and complemented presentations due to Dehornoy [19]. A complement on a set A is a partial mapping $f : A \times A \to A^*$, that is to the set of words on A; such that $f(x, x) = 1$ for all $x \in A$, and $f(x, y)$ is defined whenever $f(y, x)$ is. In particular we call a complement, f, complete, if $f(x, y)$ exists for all $x, y \in A$.

Given a Bessis group of rank 3, G, we can choose a complete complement, f, on the atoms of G, $A = \{(x_1.g)N \mid g \in B_n\}$, such that $af(a, b) = bf(b, a)$ for all pairs of atoms $a, b \in A$, since we have $a = (x_1.g)N$ and $b = (x_1.h)N$ for some $g, h \in B_3$ and therefore $a(x_2.gx_3.g)N = b(x_2hx_3h)N = \Delta$. In particular for each Bessis group of rank 3 we choose a complement f_G, satisfying $af_G(a, b) = bf_G(b, a)$ for all $a, b \in A$; and such that $f_G(a, b)$ is of minimal length, that is for all pairs of words $a', b' \in A*$ such that $aa' = bb'$ we have $l(f_G(a, b)) = l(f_G(b, a)) \leq l(a') = l(b')$.

Using this complement f_G, each Bessis group has a complemented presentation:

$$G = \langle A \mid af_G(a, b) = bf_G(b, a), \text{for all } a, b \in A\rangle$$

We define a rewriting process $\curvearrowright$ to be the reflexive-transitive closure of:

$$\{ua^{-1}bv \to uf_G(a, b)f_G(b, a)^{-1}v \mid a, b \in A, u, v \in (A \cup A^{-1})^*\}$$

If $u \in (A \cup A^{-1})^*$, then we define u^+ to be the unique $v \in A^*$ such that $u \curvearrowright vw^{-1}$ for some $w \in A^*$ if such v, w exist. Such a u^+ is unique since $\curvearrowright$ is a complete rewriting system.

We are now ready to define the under operation on $A^* = G^+$, for a Bessis group G.

Definition 3.1.4 *Given $u, v \in G^+$ we define $u \setminus v := (u^{-1}v)^+$*

In particular for $a, b \in A$ we have $a^{-1}b = f_G(a, b)f_G(b, a)^{-1}$, so that $a \setminus b = f_G(a, b)$.

Definition 3.1.5 *In a Bessis group G, with set of atoms A, a triple $(a, b, c) \in A^3$ satisfies the cube condition if we have:*

$$(a \setminus b) \setminus (a \setminus c) = (b \setminus a) \setminus (b \setminus c)$$

The following lemma can be found as proposition 4.16 in [18]:

Lemma 3.1.6 *Let $G = F_n/N_H$ be the Bessis group associated to a subgroup H of B_n, then (G, Δ) is a Bessis-Garside pair if and only if any triple of atoms in G satisfies the cube condition, that is:*

$$(a \setminus b) \setminus (a \setminus c) = (b \setminus a) \setminus (b \setminus c) \text{ for all } (a, b, c) \in A_H^3$$

We can present the cube condition on a triple diagramatically as shown in 3.1, which shows a subgraph of the Cayley graph of a Bessis-Garside group. Here we consider all orderings of the triple $(p, q, r) \in A^3$ in the cube condition and see that the condition being satisfied for them all implies that the set of atoms $\{p, q, r\}$ has a least upper bound or meet with respect to the left divisibility ordering.

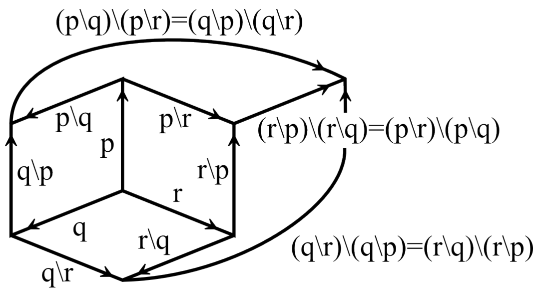

Figure 3.1: The cube condition for 3 atoms p, q, r

3.2 Rank 2 Bessis-Garside groups

In this section we present the theory of Bessis, and Bessis-Garside, groups of rank two, as an introductory example for what follows (namely the more convoluted rank three case).

3.2.1 Rank 2 Bessis groups are Garside

Since $B_2 \cong \mathbb{Z}$, the subgroups of B_2 can easily be listed, we call them: $H_k = \langle \sigma_1^k \rangle$. Associated to each of these subgroups of B_2 is a normal subgroup of F_2 as described above, namely N_k, the normal subgroup generated by all $\{x(xh)^{-1} \mid x \in F_2, h \in H_k\}$.

For the letters a, b, and for $k \in \mathbb{N}$, we use the notation $(a, b; k)$ to denote the product $aba \cdots$ of length k. We put $(a, b; -k) = (b^{-1}, a^{-1}; k)$.

Lemma 3.2.1

$$x_1.\sigma_1^k = (x_2^{-1}, x_1^{-1}; k-1)(x_2^{-1}, x_1^{-1}; k)^{-1}$$
$$x_2.\sigma_1^k = (x_2^{-1}, x_1^{-1}; k)(x_2^{-1}, x_1^{-1}; k+1)^{-1}$$

Proof. First note that $x_1x_2.\sigma_1 = x_1x_2$, and so $x_2^{-1}x_1^{-1}.\sigma_1 = x_2^{-1}x_1^{-1}$. Now $x_1.\sigma_1^0 = x_1$, $x_1.\sigma_1^1 = x_2$, so the first expression of the lemma holds for $k = 0, 1$. Now assume that the first expression of the lemma holds for k, then:

$$
\begin{aligned}
x_1\sigma_1^{k+1} &= ((x_2^{-1}, x_1^{-1}; k-1)(x_2^{-1}, x_1^{-1}; k)^{-1}).\sigma_1 \\
&= \begin{cases} (x_2^{-1}, x_1^{-1}; k-1)(x_2.\sigma_1)(x_2^{-1}, x_1^{-1}; k-1)^{-1} & \text{if } k \text{ is odd;} \\ (x_2^{-1}, x_1^{-1}; k-2)(x_2^{-1}.\sigma_1)(x_2^{-1}, x_1^{-1}; k)^{-1} & \text{if } k \text{ is even.} \end{cases} \\
&= (x_2^{-1}, x_1^{-1}; k)(x_2^{-1}, x_1^{-1}; k+1)^{-1}
\end{aligned}
$$

so by induction the result holds for $x_1.\sigma_1^k$. The expression of $x_2.\sigma_1^k$, holds since $x_1.\sigma_1 = x_2$, so that $x_2.\sigma_1^k = x_1.\sigma_1^{k+1}$. ■

Recall that the Artin groups of rank 2 are $I_2(k) = \langle a, b \mid (a, b; k) = (b, a; k)\rangle$, and $I_2(\infty) = F_2$.

Corollary 3.2.2 *For all $k \in \mathbb{N}_{\geq 2}$, the rank 2 Bessis group $G_k = F_2/N_k$ is isomorphic to the Artin group $I_2(k)$. Furthermore G_0 is isomorphic to $I_2(\infty)$. That is, the Bessis groups of rank 2 are exactly the Artin groups of rank 2.*

Proof. It is apparent from this lemma 3.2.1 that $N_k = \langle (x_1(x_2^{-1}, x_1^{-1}; k)(x_2^{-1}, x_1^{-1}; k-1)^{-1}), (x_2(x_2^{-1}, x_1^{-1}; k+1)(x_2^{-1}, x_1^{-1}; k)^{-1})\rangle$. So:

$$G_k = \langle x_1, x_2 \mid (x_1, x_2; k) = (x_2, x_1; k)\rangle$$

■

From [4] we know that all rank 2 Artin groups admit a Garside structure so we have the following corollary.

Corollary 3.2.3 *All rank 2 Bessis groups are Garside, that is for all $k \in \mathbb{N}_{\geq 2}$, the pair (G_k, Δ), (where $\Delta = x_1x_2$), is a Bessis-Garside pair.*

Proof. We need only check that the cube condition holds for all triples of atoms.
The set of atoms in G_k is $A_k := \{x_1\sigma_1^m N_k \mid m \in \mathbb{Z}\}$. Modulo N_k, $x_1\sigma_1^k = x_1$ and $x_1\sigma_1^{-m} = x_1\sigma_1^{k-m}$ for $m \leq k$ so:

$$
\begin{aligned}
A_k &= \{x_1\sigma_1^m \mid m = 0, \ldots, k-1\} \\
&= \{(x_2^{-1}, x_1^{-1}; m-1)(x_2^{-1}, x_1^{-1}; m)^{-1} \mid m = 0, \ldots, k-1\}
\end{aligned}
$$

Let $a = x_1\sigma_1^p$, $b = x_1\sigma_1^q$, and $c = x_1\sigma_1^r$; and assume $p < q < r$.

$$
\begin{aligned}
a \setminus b = a \setminus c = x_1\sigma_1^{p+1} \\
b \setminus c = b \setminus a = x_1\sigma_1^{q+1} \\
c \setminus a = c \setminus b = x_1\sigma_1^{r+1}
\end{aligned}
$$

We have therefore:

$$(a \setminus b) \setminus (a \setminus c) = (a \setminus c) \setminus (a \setminus b) = 1$$
$$(b \setminus a) \setminus (b \setminus c) = (b \setminus c) \setminus (b \setminus a) = 1$$
$$(c \setminus a) \setminus (c \setminus b) = (c \setminus b) \setminus (c \setminus a) = 1$$

and thus the cube condition holds for all triples of atoms, and the result follows from lemmas 3.1.3 and 3.1.6.

■

3.2.2 Representations over finite rings

We show that Bessis-Garside of rank 2, or equivalently Artin groups of rank 2, admit representations over cyclic groups (considered as rings).

Choose some $x \in \mathbb{Z}$. We define a map, f_x, from $\mathbb{Z}$ to itself by:

$$f_x(k) = \begin{cases} x & \text{if } k \in 2\mathbb{Z}; \\ 1 & \text{otherwise.} \end{cases}$$

and a sequence $\{w_k\}$ by $w_0 = 0$, $w_1 = 1$, and $w_{k+1} = w_{k-1} + f_x(k)w_k$, so that $w_2 = 1$, $w_3 = x + 1$, $w_4 = x + 2$, $w_5 = x^2 + 3x + 1$, and so on.

Theorem 3.2.4 *Let $a, b \in \mathbb{Z}$, and define matrices A, B as:*

$$A := \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \qquad B := \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

and put $x := ab$. Then there exists a unique representation $\rho_{a,b} : G_k \to Gl(2, \mathbb{Z}_{w_k})$ given by $x_1 \mapsto A \bmod w_k$ and $x_2 \mapsto B \bmod w_k$

Proof. We first note that for k, we have:

(3.1) $w_{k-1}^2 - abw_{k-1}w_{k-2} - abw_{k-2}^2 = 1$

(3.2) $w_{k-1}^2 + abw_{k-1}w_k - abw_k^2 = 1$

which hold since $w_1^2 - abw_1w_0 - abw_0^2 = 1$ and

$$\begin{aligned} w_{k-1}^2 + abw_{k-1}w_k - abw_k^2 &= w_{k-1}(w_{k-1} + abw_{k-1} + abw_{k-2}) - ab(w_{k-1} + w_{k-2})^2 \\ &= w_{k-1}^2 - abw_{k-1}w_{k-2} - abw_{k-2}^2 \\ &= (w_{k-3} + abw_{k-2})^2 - ab(w_{k-3} + abw_{k-2})w_{k-2} - abw_{k-2}^2 \\ &= w_{k-3}^2 + abw_{k-3}w_{k-2} - abw_{k-2}^2. \end{aligned}$$

We note that:

$$ABA^{-1} := \begin{pmatrix} 1-ab & b \\ -a^2b & 1+ab \end{pmatrix} \qquad ABAB^{-1}A^{-1} := \begin{pmatrix} 1+ab+a^2b^2 & -ab^2 \\ a+2a^2b+a^3b^2 & 1-ab-a^2b^2 \end{pmatrix}$$

that is for $k = 2, 3$:

$$(3.3)\ \ (A, B; k)(A, B; k-1)^{-1} = \begin{pmatrix} 1+(-1)^{k+1}w_{k-1}w_k ab & b+(-1)^k w_{k-2}w_k f_k b \\ (-1)^{k+1}w_k^2 f_k a & 1+(-1)^k w_{k-1}w_k ab \end{pmatrix}$$

and therefore $(A, B; k)(A, B; k-1)^{-1} = B(\text{mod } w_k)$ for $k = 2, 3$.

We use that $(A, B; k)(A, B; k-1)^{-1} = AB((A, B; k-1)(A, B; k-2)^{-1})^{-1}$ to prove (3.3) for all k by induction on k.
Suppose k is even, then using (3.1), and (3.2):

$$\begin{aligned}
[(A, B; k)(A, B; k-1)^{-1}]_{1,1} &= 1 - w_{k-2}w_{k-1}ab - w_{k-1}^2(ab) \\
&= 1 - w_{k-1}ab(w_{k-2} + w_{k-1}) \\
&= 1 - w_{k-1}w_k ab
\end{aligned}$$

$$\begin{aligned}
[(A, B; k)(A, B; k-1)^{-1}]_{1,2} &= -b + bw_{k-3}w_{k-1} + b + ab^2 w_{k-2}w_{k-1} \\
&= b(w_{k-3}w_{k-1} + ab(w_{k-2}w_k - w_{k-2}^2)) \\
&= ab^2 w_{k-2}w_k + b(w_{k-3}w_{k-1} - abw_{k-2}^2) \\
&= ab^2 w_{k-2}w_k + b(w_{k-3}w_{k-1} - w_{k-2}w_{k-1} + w_{k-3}w_{k-2}) \\
&= ab^2 w_{k-2}w_k + b(abw_{k-3}w_{k-2} + w_{k-3}^2 - abw_{k-2}^2) \\
&= b + ab^2 w_{k-2}w_k
\end{aligned}$$

$$\begin{aligned}
[(A, B; k)(A, B; k-1)^{-1}]_{2,1} &= a - a^2bw_{k-2}w_{k-1} - aw_{k-1}^2 - a^2bw_{k-1}^2 \\
&= a(1 - abw_{k-2}w_{k-1} - w_{k-1}^2 - abw_{k-1}^2) \\
&= a(-abw_k^2 + abw_{k-1}w_k - abw_{k-2}w_{k-1} - abw_{k-1}^2) \\
&= a(-abw_k + ab(w_{k-1}(w_{k-2} + w_{k-1}) - w_{k-1}^2 - w_{k-2}w_{k-1})) \\
&= -a^2bw_k^2
\end{aligned}$$

$$\begin{aligned}
[(A, B; k)(A, B; k-1)^{-1}]_{2,2} &= -ab + abw_{k-3}w_{k-1} + 1 + ab + abw_{k-2}w_{k-1} + (ab)^2 w_{k-2}w_{k-1} \\
&= 1 + ab(w_{k-1}(w_{k-3} + w_{k-2} + abw_{k-2})) \\
&= 1 + ab(w_{k-1}(w_{k-2} + w_{k-1})) \\
&= 1 + abw_{k-1}w_k
\end{aligned}$$

and:

$$\begin{aligned}[(A,B;k+1)(A,B;k)^{-1}]_{1,1} &= 1 + w_{k-1}w_k ab + w_k^2(ab)^2 \\ &= 1 + ab(w_k(w_{k-1}+abw_k)) \\ &= 1 + abw_k w_{k+1}\end{aligned}$$

$$\begin{aligned}[(A,B;k+1)(A,B;k)^{-1}]_{1,2} &= -b - ab^2 w_{k-2}w_{k-1} + b - ab^2 w_{k-1}w_k \\ &= bw_{k-2}(w_{k-1} - w_{k+1}) + bw_{k-1}(w_{k-1} - w_{k+1}) \\ &= -bw_{k-1}w_{k+1} + b(w_{k-2}w_{k-1} - w_{k-2}w_{k+1} + w_{k-1}^2) \\ &= -bw_{k-1}w_{k+1} + b(-abw_{k-2}^2 - abw_{k-2}w_{k-1} + w_{k-1}^2) \\ &= b - bw_{k-1}w_{k+1}\end{aligned}$$

$$\begin{aligned}[(A,B;k+1)(A,B;k)^{-1}]_{2,1} &= a + a^2bw_{k-1}w_k + a^2bw_k^2 + a^3b^2w_k^2 \\ &= a(1 + abw_k^2 + abw_{k-1}w_k + (ab)^2w_k^2) \\ &= a(w_{k+1}^2 - abw_kw_{k+1} + abw_{k-1}w_k + (ab)^2w_k^2) \\ &= a(w_{k+1}^2 + ab(abw_k^2 + w_{k-1}w_k - abw_k^2 - w_{k-1}w_k)) \\ &= aw_{k+1}^2\end{aligned}$$

$$\begin{aligned}[(A,B;k+1)(A,B;k)^{-1}]_{2,2} &= -ab - (ab)^2w_{k-2}w_k + 1 - abw_{k-1}w_k + ab - (ab)^2w_{k-1}w_k \\ &= 1 - ab(abw_{k-2}w_k + w_{k-1}w_k + abw_{k-1}w_k) \\ &= 1 - ab(x_kw_{k-1} + abw_k(w_{k-2} + w_{k-1})) \\ &= 1 - ab(x_k(w_{k-1} + abw_k)) \\ &= 1 - abw_kw_{k+1}\end{aligned}$$

and the result holds.

■

3.3 Cycle presentations

In this section we introduce the notation of cycles, and define cycle presentations, which will prove a useful tool for understanding Bessis groups of higher rank. We apply this terminology to the rank 2 Bessis groups shown earlier in the chapter.

Definition 3.3.1 *Given an ordered k-tuple over a set A, $C = (c_1, c_2, \ldots, c_k) \in A^k$, the* ***cycle of length*** k*, $K(C)$, associated to this k-tuple is the set of equalities $K(C) := \{c_kc_1 = c_ic_{i+1} \mid i = 1 \ldots k-1\}$. We also allow cycles of infinite length, where $C = (\ldots, c_{-1}, c_0, c_1, \ldots)$ with $c_i \in A$, the cycle associated to C is $K(C) = \{c_ic_{i+1} = c_{i+1}c_{i+2} \mid i \in$*

$\mathbb{Z}\}$. *We will may use the k-tuple, or sequence C as a notational shorthand for the associated cycle $K(C)$.*

So for an ordered set of letters, the associated cycle is a set of homogenous relations of length 2 on those letters. Of particular interest throughout the remainder of this thesis will be groups where all relations come from cycles.

Definition 3.3.2 *Consider a pair (A, C) where A is some set, and C is a collection of tuples over A indexed by some (possibly infinite set) I, that is $C = \{C_i \mid i \in I\}$, and let k_i be the length of the tuple C_i. Suppose also that $C_i \cap C_j \leq 1$ for $i, j \in I$ and $i \neq j$. Then the* ***cycle presentation*** *associated to this pair is the presentation with set of generators A, and the set of relations $R = \bigcup_{i \in I} K(C_i)$.*

We say that a group G is ***cycle presented*** *if there exists a pair (A, C) such that $G = \left\langle A \mid \bigcup_{i \in I} K(C_i) \right\rangle$.*

In such a group, we denote $e(C_i)$, and call the **evaluation of a cycle**, the value of the product of adjacent elements in the tuple C_i, since the relations of the associated cycle hold in the group, this is a well defined map from C to G.

We can associate to each cycle presentation a directed graph, as follows.

Definition 3.3.3 *Let (A, C) define a cycle presention, then the* ***cycle graph*** *of (A, C) has vertex set A, and edges $E = \{(x, y) \in A^2 \mid \exists C_i = (\ldots, x, y, \ldots) \in C\}$.*

We can now give an interpretation of the theory of Bessis-Garside groups of rank 2 in terms of cycle presentations.

Proposition 3.3.4 *For all $k \in \mathbb{N}_{k \geq 2}$, the Bessis group G_k admits a cycle presentation with atoms A_k and one cycle, namely:*

$$(x_1.\sigma_1^0, x_1.\sigma_1^1, \ldots, x_1.\sigma_1^{k-2}, x_1.\sigma_1^{k-1})$$

Proof. Clearly the atoms A_k generate G_k. We show that all the relations given by the cycle hold in G_k. The relations $x_1.\sigma_1^i x_1.\sigma_1^{i+1} = x_1.\sigma_1^{i+1} x_1.\sigma_1^{i+2}$ for $i = 0, \ldots, k-3$, follow from $x_1 x_1^{-1} = x_1^{-1} x_1 = x_2 x_2^{-1} = x_2^{-1} x_2 = 1$. Finally we have $x_1.\sigma_1^{k-1} x_1.\sigma_1^0 = x_1.\sigma_1^0 x_1.\sigma_1^1$, that is:

$$\begin{aligned}
x_1.\sigma_1^{k-1} x_1.\sigma_1^0 &= (x_2^{-1}, x_1^{-1}; k-2)(x_2^{-1}, x_1^{-1}; k-1)^{-1} x_1 \\
&= (x_2^{-1}, x_1^{-1}; k-2)(x_1^{-1}, x_2^{-1}; k)^{-1} \\
&= (x_2^{-1}, x_1^{-1}; k-2)(x_2^{-1}, x_1^{-1}; k)^{-1} \\
&= (x_2^{-1} x_1^{-1})^{-1} \\
&= x_1.\sigma_1^0 x_1.\sigma_1^1
\end{aligned}$$

Conversely the sole relation of G_k is $(x_1, x_2; k) = (x_2, x_1; k)$ which can see in the immediately preceding lines holds if and only if $x_1.\sigma_1^{k-1} x_1.\sigma_1^0 = x_1.\sigma_1^0 x_1.\sigma_1^1$. This completes the proof. ■

This is essentially to say that a cycle presented group with exactly one cycle of length k, and with all generators present in that cycle is necessarily $I_2(k)$.

Now that we have an example of a cycle presented group, we can give the associated cycle graph, see figure 3.2

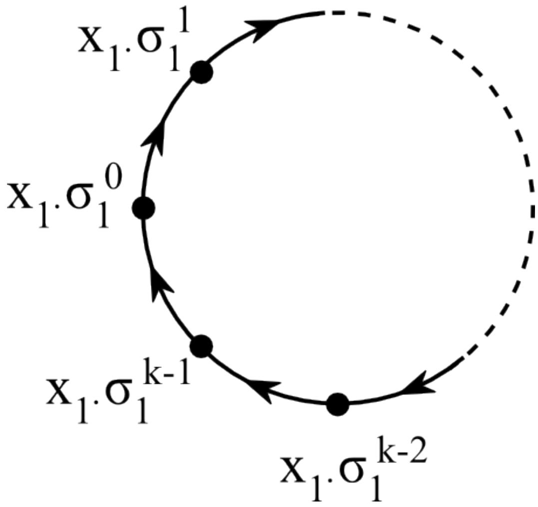

Figure 3.2: $\Gamma(I_2(k))$

Chapter 4

Bessis-Garside quotients of spherical and affine Artin groups of rank 3

It is known from [4] that all Artin groups of spherical type are Garside, and from [20] and [21] that among the Artin groups of affine type $\tilde{A}$ and $\tilde{C}$ are Garside. We can apply lemma 4.2.2 in the following to see that these Garsde structures can be seen as Bessis-Garside structures in the rank 3 cases. In this chapter we begin further study Bessis-Garside groups of rank 3, in particular focusing on the Artin groups of affine type and their quotients.

In the first section we give an example of a rank 3 Bessis groups which is not Garside.

In the second section we introduce a family of cycle presentations, which we then show to present groups that are Bessis, are Bessis-Garside, and are in fact all Bessis-Garside quotients of $\tilde{C}_2$.

In section 4.3 we generalise the results of the previous section to give necessary and sufficient conditions for a cycle presentation to describe a Bessis-Garside group of rank 3.

We then reproduce the results of section 4.2, with respect to the affine Artin group $\tilde{A}_2$, and for the following group:

$$\tilde{H}_2 = \left\langle a, b, c \mid \begin{array}{c} ababab = bababa \\ bcb = cbc, ac = ca \end{array} \right\rangle$$

(although in the latter case we find that $\tilde{H}_2$ is not itself Bessis-Garside).

We then go on to give cycle presentations for the spherical Artin groups of rank 3, and in the final section we give representations for some Bessis-Garside groups of rank 3 over cyclic groups.

4.1 Non-Garside Bessis groups of rank 3

Definition 4.1.1 *Let G be the Bessis group of rank 3 associated to the subgroup H of B_3. We call (x_1, x_2, x_3) the standard triple. We say the standard triple is regular if for any $(y_1, y_2, y_3) = (x_1, x_2, x_3).g$ with $g \in B_n$ we have $y_1 N \neq y_2 N$.*

Lemma 4.1.2 *Let G be a Bessis group of rank 3, with atoms A, and $\Delta := x_1 x_2 x_3$. Then for any triple $(p, q, r) \in A^3$, $pqr = \Delta$, and $p \neq q$ implies $p \setminus q = q$.*

Proof. Since $p \in A$, $p = x_1.g$ for some $g \in B_3$. $(p, q, r).\sigma_1^2 = (q^{-1}pq, q^{-1}p^{-1}qpq, r)$ so $q^{-1}pq \in A$, and thus $pq = q(q^{-1}pq)$ and $p \setminus q = q$. ■

Example 4.1.3 *Let H be the subgroup of B_3 generated by $\sigma_1^{-1}\sigma_2^{-1}\sigma_1$. Then we can give all relations in G_H which is generated by $x_1 N = a, x_2 N = b, x_3 N = c$:*

$$\begin{aligned}
a(a.\sigma_1^{-1}\sigma_2^{-1}\sigma_1)^{-1} &= a(abcb^{-1}a^{-1})^{-1} \in N \\
b(b.\sigma_1^{-1}\sigma_2^{-1}\sigma_1)^{-1} &= b(b)^{-1} \in N \\
c(c.\sigma_1^{-1}\sigma_2^{-1}\sigma_1)^{-1} &= c(b^{-1}ab)^{-1} \in N
\end{aligned}$$

that is $c = b^{-1}ab \in G$

From lemma 4.1.2 we know that $a \setminus b = b$, and $a \setminus c = b \setminus c = c$. Furthermore since $ab = b(b^{-1}ab)$, we have $b \setminus a = b^{-1}ab$. So let us consider the cube condition with respect to the triple $(a, b, c) \in A^3$:

$$\begin{aligned}
(a \setminus b) \setminus (a \setminus c) &= b \setminus c && = c \\
(b \setminus a) \setminus (b \setminus c) &= b^{-1}ab \setminus c && = c \setminus c = 1
\end{aligned}$$

So the Bessis group associated to H is not Garside as it fails to satisfy the cube condition for the triple (a, b, c).

In fact this example reveals a general result showing that a certain class of Bessis groups cannot be Garside.

We must first introduce some notation. Suppose G is a group given by a cycle presentation, with atoms A. We let the braid group on 3 strands, B_3 act on the set of triples over A in the normal manner, that is:

$$\begin{aligned}
(p, q, r).\sigma_1 &= (q, q^{-1}pq, r) \\
(p, q, r).\sigma_2 &= (p, r, r^{-1}qr)
\end{aligned}$$

we note that this action preserves the product pqr, also that this action (called the Hurwitz action) on triples is distinct from the Artin action of B_3 on the free group F_3.

Proposition 4.1.4 *Let G be a Bessis group of rank 3, with $\Delta = x_1x_2x_3$, such that there exists a triple of atoms $(y_1, y_2, y_3) = (x_1, x_2, x_3).g$ for some $g \in G$, with $y_2N = y_3N$. That is G a Bessis group whose standard triple is not regular. If G is Garside, then $G = \mathbb{Z}$.*

Proof. Consider $(y_1, y_2, y_3).\sigma_1^{-2} = (y_1y_2y_1y_2^{-1}y_1^{-1}, y_1y_2y_1^{-1}, y_3)$. We have:

$$\begin{aligned} ((y_1y_2y_1^{-1}) \setminus y_1) \setminus ((y_1y_2y_1^{-1}) \setminus y_2) &= y_1 \setminus y_2 && = y_2 \\ (y_1 \setminus (y_1y_2y_1^{-1})) \setminus (y_1 \setminus y_2) &= y_2 \setminus y_2 && = 1 \end{aligned}$$

So if G is Garside we must have $y_1 \setminus y_2 = 1$ so that $y_1N = y_2N$. This implies that $A = \{x_1.hN \mid h \in B_n\} = \{y_1.g^{-1}hN \mid h \in B_n\} = \{y_1\}$, and therefore that $G = \mathbb{Z}$. ∎

We will see in section 4.5 an example of a Bessis group with regular standard triple which is not Garside, namely the affine Artin group $\tilde{H}_2$.

4.2 An infinite family of cycle presented groups

4.2.1 Definition

We introduce a family of groups $\tilde{C}_2(k)$, $k \geq 1$, with cycle presentations.

Definition 4.2.1 *We denote by $\tilde{C}_2(k)$ the cycle presented group, with set of generators:*

$$A_k := \{a_m, b_l, c_l^m \mid l \in \mathbb{Z}_{k+2}, m \in \{0,1\}\}$$

and relations given by cycles:

$$\begin{aligned} &K(c_0^m, c_1^m, \ldots, c_{k+1}^m), && m \in \{0,1\} \\ &K(c_l^1, c_{l+1}^0), && l \in \mathbb{Z}_{k+2} \\ &K(a_{|m-1|}, c_{l-1+2m}^{|m-1|}, b_{l+m}, c_{l+1}^m), && m \in \{0,1\},\ l \in \mathbb{Z}_{k+2}. \end{aligned}$$

Let $\Delta = c_0^1 a_0 c_1^0$, then we have the following factorisations of Δ:

$$\begin{aligned} \Delta &= a_m c_l^m c_{l+1}^m \\ &= b_l c_l^1 c_{l+1}^0 \\ &= c_l^m a_{|m-1|} c_{l-1+2m}^{|m-1|} = c_l^m c_{l-1+2m}^{|m-1|} b_{l+m} \\ &= c_l^m b_{l+m} c_{l+1}^m = c_l^m c_{l+1}^m a_{|m-1|} \end{aligned}$$

or equivalently:

(4.1)

$$\begin{aligned}\Delta &= a_m e(c_0^m, c_1^m, \dots, c_{k+1}^m), m \in \{0,1\}\\ &= b_l e(c_l^1, c_{l+1}^0), l \in \mathbb{Z}_{k+2}\\ &= c_l^m e(a_{|m-1|}, c_{l-1+2m}^{|m-1|}, b_{l+m}, c_{l+1}^m), m \in \{0,1\},\ l \in \mathbb{Z}_{k+2}.\end{aligned}$$

Similar calculations show:

(4.2)

$$\begin{aligned}\Delta &= e(c_0^m, c_1^m, \dots, c_{k+1}^m) a_{|m-1|}, m \in \{0,1\}\\ &= e(c_l^1, c_{l+1}^0) b_{l+1}, l \in \mathbb{Z}_{k+2}\\ &= e(a_{|m-1|}, c_{l-1+2m}^{|m-1|}, b_{l+m}, c_{l+1}^m) c_{l+2m}^{|m-1|}, m \in \{0,1\},\ l \in \mathbb{Z}_{k+2}.\end{aligned}$$

Having given presentations for these groups, we will show over the next two subsections that they form an infinite family of Bessis-Garside groups of rank 3, with Garside element Δ. The proofs that these groups satisfy the necessary properties will be extended in the remainder of the chapter to provide a set of necessary and sufficient conditions for cycle presented groups to be Bessis-Garside of rank 3.

4.2.2 $\tilde{C}_2(k)$ are Bessis

Suppose G is a group given by a cycle presentation, with atoms A. Then we call $p_j : A^3 \to A$ the projection to the j-th coordinate.

Lemma 4.2.2 *Let G be a cycle presented group, with atoms A, and cycles C. If there exists a triple of atoms $(a, b, c) \in A^3$ satisfying $p_1((a, b, c).B_3) = A$, and for every cycle $K \in C$ there exists an atom $x \in A$ such that $xe(K) = abc$; then G is a Bessis group of rank 3*

Proof. Since $p_1((a, b, c).B_3) = A$, and clearly $p_1((a, b, c).B_3) \subset \langle a, b, c\rangle$, we can see that G is generated by $\{a, b, c\}$.

Now let H be the subgroup of B_3 consisting of all braids which fix the triple (a, b, c) in A^3, on which B_3 acts as given above, that is:

$$H := \{g \in B_3 \mid (a, b, c).g = (a, b, c)\}.$$

We then consider the Bessis group of rank 3, G_H associated to this subgroup, and we claim that $G_H \cong G$, with isomorphism given by $x_1 \mapsto a, x_2 \mapsto b, x_3 \mapsto c$.

The above map is then a homomorphism since all relations in G_H are of the form $x_i(x_i.g)^{-1} = 1$ with $g \in H$, which maps to $(p_i(a, b, c))(p_i((a, b, c).g))^{-1} = (p_i(a, b, c))(p_i(a, b, c))^{-1} = 1$ as required, using the definition of H. So the map from G_H to G is a homomorphism, and is surjective since G is generated by a, b and c. Given some cycle K in C, we have $x \in A$ such that $xe(K) = abc$. Suppose $x = p_1((a,b,c).g)$ and $l(K) = j_K$, then:

$$(a,b,c).g\sigma_2^{j_K}g^{-1} = (a,b,c)$$

in the group G, so $g\sigma_2^{j_K}g^{-1} \in H$. So $x_i.g\sigma_2^{j_K}g^{-1} = x_i$ and in particular:

$$x_2.(g\sigma_2^{j_K-k})x_3.(g\sigma_2^{j_K-k}) = x_2.(g\sigma_2^k)x_3.(g\sigma_2^k)$$

for $k = 0, 1, \ldots, j_k$, that is all of the cycle relations given by K in G hold also in G_H (when written in terms of x_1, x_2, x_3 rather than a, b, c), so the inverse map is a (surjective) homomorphism, and the groups are isomorphic.

■

Lemma 4.2.3 *For $k \geq 1$ the group $\tilde{C_2}(k)$ is a Bessis group of rank 3.*

Proof. Consider the triple $(c_0^1, a_0, c_1^0) \in A^3$, then we have

$$(c_0^1, a_0, c_1^0).\sigma_2\sigma_2\sigma_1 = (b_1, c_2^0, c_1^1)$$

$$(b_1, c_2^0, c_1^1).(\sigma_1\sigma_2\sigma_2\sigma_1\sigma_2)^{i-1} = (b_i, c_{i+1}^0, c_i^1)$$

which clearly shows that:

$$(b_i, c_{i+1}^0, c_i^1).\sigma_1 = (c_{i+1}^0, a_1, c_i^1)$$

$$(b_i, c_{i+1}^0, c_i^1).\sigma_2\sigma_1 = (c_i^1, a_0, c_{i+1}^0)$$

and finally:

$$(c_0^1, a_0, c_1^0).\sigma_1\sigma_1\sigma_2\sigma_2\sigma_1 = (a_1, c_k^1, c_{k+1}^1).$$

so that:

$$\begin{aligned}
p_1((c_0^1, a_0, c_1^0).\sigma_1) &= a_0 \\
p_1((c_0^1, a_0, c_1^0).\sigma_1\sigma_1\sigma_2\sigma_2\sigma_1) &= a_1 \\
p_1((c_0^1, a_0, c_1^0).\sigma_1\sigma_1\sigma_2\sigma_2\sigma_1(\sigma_1\sigma_2\sigma_2\sigma_1\sigma_2)^{i-1}) &= b_i \\
p_1((c_0^1, a_0, c_1^0).\sigma_1\sigma_1\sigma_2\sigma_2\sigma_1(\sigma_1\sigma_2\sigma_2\sigma_1\sigma_2)^{i-1}\sigma_1) &= c_{i+1}^0 \\
p_1((c_0^1, a_0, c_1^0).\sigma_1\sigma_1\sigma_2\sigma_2\sigma_1(\sigma_1\sigma_2\sigma_2\sigma_1\sigma_2)^{i-1}\sigma_2\sigma_1) &= c_i^1
\end{aligned}$$

for $i \in \mathbb{Z}_{k+2}$. This shows that $p_1((c_0^1, a_0, c_1^0).B_3) = A$.

From (4.1), we can see that for any cycle K of $\tilde{C_2(k)}$, there exists an atom x such that $xe(K) = c_0^1 a_0 c_1^0$. So the result holds by lemma 4.2.2

■

4.2.3 The subgroups of B_3 associated to $\tilde{C_2}(k)$

In this subsection we give an explicit description of $\tilde{C_2}(1)$ as a Bessis group, which justifies a theorem on the subgroups of B_3 associated to the Bessis groups $\tilde{C_2}(k)$ for $k \geq 1$. The proof of that theorem follows from a fact about the groups $\tilde{C_2}(k)$ which is shown in a following subsection.

Lemma 4.2.4 *The group $\tilde{C_2}(1)$ is isomorphic to the group given by the presentation:*

$$G = \left\langle a, b, c \mid \begin{array}{l} abab = baba, bcbc = cbcb, ac = ca \\ b^{-1}abcb^{-1}ab = cb^{-1}abc \end{array} \right\rangle$$

Proof. We recall from the proof of Lemma 4.2.3 that the atoms c_0^1, a_0, c_1^0 generate $\tilde{C_2}(1)$, so we may define a map ϕ from $\tilde{C_2}(1)$ to the group G defined in the lemma by:

$$\begin{aligned} \phi : \tilde{C_2}(k) &\to G \\ c_0^1 &\mapsto a \\ a_0 &\mapsto b \\ c_1^0 &\mapsto c \end{aligned}$$

We can then check that this map is in fact a homomorphism. First we give words representing the images of each of the atoms of $\tilde{C_2}(k)$:

$$\begin{aligned} \phi(a_1) &= c^{-1}aba^{-1}c \\ \phi(b_0) &= b^{-1}a^{-1}bab \\ \phi(b_1) &= c^{-1}bc \\ \phi(b_2) &= c^{-1}b^{-1}a^{-1}c^{-1}bcabc \\ \phi(c_0^0) &= b^{-1}ab \\ \phi(c_2^0) &= c^{-1}b^{-1}abc \\ \phi(c_1^1) &= c^{-1}b^{-1}cbc \\ \phi(c_2^1) &= bc^{-1}b^{-1}abcb^{-1} \end{aligned}$$

It is then a simple matter to check that the relations of each cycle hold in the image G:

$\phi(K(a_0, c_0^0, b_0, c_0^1))$:

$bb^{-1}ab = b^{-1}abb^{-1}a^{-1}bab = b^{-1}a^{-1}baba = ab$

$\phi(K(a_0, c_1^0, b_1, c_1^1))$:

$bc = cc^{-1}bc = c^{-1}bcc^{-1}b^{-1}cbc = c^{-1}b^{-1}cbcb$

$\phi(K(a_0, c_2^0, b_2, c_2^1))$:

$bc^{-1}b^{-1}abc = c^{-1}b^{-1}abcc^{-1}b^{-1}a^{-1}c^{-1}bcabc =$

$c^{-1}b^{-1}a^{-1}c^{-1}bcabcc^{-1}b^{-1}c^{-1}bcac^{-1}b^{-1}cbc = c^{-1}b^{-1}c^{-1}bcac^{-1}b^{-1}cbcb$

$\phi(K(a_1, c_0^1, b_1, c_2^1))$:

$c^{-1}aba^{-1}ca = ac^{-1}bc = c^{-1}bcc^{-1}b^{-1}abc = c^{-1}b^{-1}abcc^{-1}aba^{-1}c$

$\phi(K(a_1, c_1^1, b_2, c_0^1))$:

$c^{-1}aba^{-1}cc^{-1}b^{-1}cbc = c^{-1}b^{-1}cbcc^{-1}b^{-1}a^{-1}c^{-1}bcabc =$

$c^{-1}b^{-1}a^{-1}c^{-1}bcabcb^{-1}ab = b^{-1}abc^{-1}aba^{-1}c$

$\phi(K(a_1, c_2^1, b_0, c_1^1))$:

$c^{-1}aba^{-1}cc^{-1}b^{-1}c^{-1}bcac^{-1}b^{-1}cbc =$

$c^{-1}b^{-1}c^{-1}bcac^{-1}b^{-1}cbcb^{-1}a^{-1}bab = b^{-1}a^{-1}babc = cc^{-1}aba^{-1}c$

$\phi(K(c_0^1, c_1^0))$:

$ac = ca$

$\phi(K(c_1^1, c_2^0))$:

$c^{-1}b^{-1}cbcc^{-1}b^{-1}abc = c^{-1}b^{-1}abcc^{-1}b^{-1}cbc$

$\phi(K(c_2^1, c_0^0))$:

$b^{-1}abc^{-1}b^{-1}c^{-1}bcac^{-1}b^{-1}cbc = c^{-1}b^{-1}c^{-1}bcac^{-1}b^{-1}cbcb^{-1}ab$

$\phi(K(c_0^0, c_1^0, c_2^0))$:

$b^{-1}abc = cc^{-1}b^{-1}abc = c^{-1}b^{-1}abcb^{-1}ab$

$\phi(K(c_0^1, c_1^1, c_2^1))$:

$ac^{-1}b^{-1}cbc = c^{-1}b^{-1}cbcc^{-1}b^{-1}c^{-1}bcac^{-1}b^{-1}cbc = c^{-1}b^{-1}c^{-1}bcac^{-1}b^{-1}cbca$

Furthermore the inverse map, ϕ^{-1}, taking $a \mapsto c_0^1, b \mapsto a_0, c \mapsto c_1^0$, is also a homomorphism since the relations of the cycles (a_0, c_0^0, b_0, c_0^1), (a_0, c_1^0, b_1, c_1^1), (c_0^1, c_1^0), and (c_0^0, c_1^0, c_2^0) are equivalent to the relations $c_0^1 a_0 c_0^1 a_0 = a_0 c_0^1 a_0 c_0^1$, $a_0 c_1^0 a_0 c_1^0 = c_1^0 a_0 c_1^0 a_0$, $c_0^1 c_1^0 = c_1^0 c_0^1$, and $c_1^0 (a_0)^{-1} c_0^1 a_0 c_1^0 = (a_0)^{-1} c_0^1 a_0 c_1^0 (a_0)^{-1} c_0^1 a_0$ respectively. Since both ϕ are homomorphisms (and clearly surjective), we have $G \cong \tilde{C}_2(1)$ as stated.

■

Theorem 4.2.5 *Let H be the subgroup of B_3 generated by $\sigma_1^4, \sigma_2^4, \sigma_1^{-1}\sigma_2^2\sigma_1, \sigma_1\sigma_2^{k+2}\sigma_1^{-1}$. Then the Bessis group associated to H is $\tilde{C}_2(k)$.*

Proof. By Theorem 4.2.7 (in a following subsection), the group $\tilde{C}_2(\infty)$ with presentation:

$$\tilde{C}_2(\infty) = \left\langle \begin{array}{lr|lr} a_m, & m \in \{0,1\} & K(\ldots, c^m_{-1}, c^m_0, c^m_1, \ldots), & m \in \{0,1\}, \\ b_l, & l \in \mathbb{Z} & K(c^0_{l+1}, c^1_l), & l \in \mathbb{Z}, \\ c^m_l, & l \in \mathbb{Z},\ m \in \{0,1\} & K(a_m, c^m_l, b_{l+m}, c^{|m-1|}_{l+2m}), & l \in \mathbb{Z},\ m \in \{0,1\} \end{array} \right\rangle$$

is isomorphic to the affine Artin group $\tilde{C}_2$. $\tilde{C}_2$ has presentation:

$$\langle a, b, c \mid abab = baba, bcbc = cbcb, ac = ca \rangle$$

and is the Bessis group given by $H_{\tilde{C}_2} = \left\langle \sigma_1^4, \sigma_2^4, \sigma_1^{-1}\sigma_2^2\sigma_1 \right\rangle$ Now note that:

$$\tilde{C}_2(k) = \frac{\tilde{C}_2(\infty)}{c^0_0 = c^0_{k+2}}$$

and $c^0_0 = c^0_{k+2}$ is equivalent to $(b^{-1}ab, c; k) = (c, b^{-1}ab; k)$, which is the result of adding $\sigma_1\sigma_2^{k+2}\sigma_1^{-1}$ to the generators of H. ■

4.2.4 $\tilde{C}_2(k)$ are Garside

Lemma 4.2.6 *The pair $(\tilde{C}_2(k), \Delta)$ is a Garside pair for all $k \geq 1$.*

Proof. From lemma 4.2.3 and lemma 3.1.6 it is sufficient to establish that the cube condition holds for all triples $(a_1, a_2, a_3) \in A^3_k$. We begin by splitting all such triples into 5 types. Given a pair of distinct atoms $(a, b) \in A^2_k$ we say that that pair of atoms occurs in the cycle $C = K(c_1, c_2, \ldots, c_r)$ if there exist $i, j \in \{1, 2, \ldots, r\}$ such that $c_i = a \neq b = c_j$. If there is no cycle C such that (a, b) occurs in C then we say that the pair (a, b) are cycle-wise disjoint. Note that if a pair of atoms are cycle-wise disjoint then we have $l(a(a \setminus b)) > 2$, and in fact $a(a \setminus b) = \Delta$ since we have seen above factorisations of Δ ((4.1)) beginning with each atom. Then all triples of atoms are of one of the following types, up to re-ordering:

1. (a_1, a_2), (a_1, a_3) and (a_2, a_3) all occur in a single cycle.
2. Each pair of atoms occurs in a different cycle.
3. (a_1, a_2) occurs in one cycle, (a_1, a_3) in another, and (a_2, a_3) are cycle-wise disjoint.
4. (a_1, a_2) occurs in one cycle, and the pairs (a_1, a_3) and (a_2, a_3) are cycle-wise disjoint.
5. All pairs of atoms are cycle-wise disjoint.

Triples of each type can be shown to satisfy the cube condition as follows:

1. Since a_1, a_2 and a_3 occur in a common cycle, for $\{i, j, k\} = \{1, 2, 3\}$ we have $a_i \setminus a_j = a_i \setminus a_k$ and thus:

$$(a_i \setminus a_j) \setminus (a_i \setminus a_k) = (a_j \setminus a_i) \setminus (a_j \setminus a_k) = 1$$

that is, the cube condition holds for all such triples.

2. Given an atom $a \in A_k$, we define the **Right Adjacency Set of** a, to be

$$R(a) := \{b \in A_k \mid a \setminus b = b\}$$

that is, the set of atoms which following a in some cycle. Similarly we can define the **Left Adjacency Set of** a, to be

$$R(a) := \{b \in A_k \mid b \setminus a = a\}$$

the set of atoms preceding a in some cycle.

For $\tilde{C}_2(k)$ we can list all right adjacent sets:

$$\begin{aligned} R(a_m) &= \{c_0^m, c_1^m, \dots, c_{k+1}^m\},\ m \in \{0,1\} \\ R(b_l) &= \{c_l^1, c_{l+1}^0\}, l \in \mathbb{Z}_{k+2} \\ R(c_l^m) &= \{a_{|m-1|}, c_{l-1+2m}^{|m-1|}, b_{l+m}, c_{l+1}^m\}, m \in \{0,1\},\ l \in \mathbb{Z}_{k+2}. \end{aligned}$$

and note that each $R(a)$ is the underlying set of some cycle which we denote C_a, and furthermore from (4.1), $\Delta = a \cdot e(C_a)$ for all $a \in A_k$. It is possible to show in a similar way that each $L(a)$ is the underlying set of some cycle which we denote ${}_aC$.

For $\{i,j,k\} = \{1,2,3\}$ the pair $(a_i \setminus a_j, a_i \setminus a_k)$ occurs in the cycle C_{a_i}, and similarly the pair $(a_j \setminus a_i, a_j \setminus a_k)$ occurs in the cycle C_{a_j} We can therefore write:

$$\begin{aligned} \Delta &= a_i(a_i \setminus a_j)((a_i \setminus a_j) \setminus (a_i \setminus a_k)) \\ &= a_j(a_j \setminus a_i)((a_j \setminus a_i) \setminus (a_j \setminus a_k)) \end{aligned}$$

which gives:

$$\begin{aligned} (a_i \setminus a_j) \setminus (a_i \setminus a_k) &= (a_i(a_i \setminus a_j))^{-1}\Delta \\ &= (a_j(a_j \setminus a_i))^{-1}\Delta \\ &= (a_j \setminus a_i) \setminus (a_j \setminus a_k) \end{aligned}$$

and so the cube condition is satisfied for triples of this type.

3. Much as in the previous case we have that $(a_1 \setminus a_2, a_1 \setminus a_3)$ occurs in the cycle C_{a_1}. Putting $\{i,j\} = \{2,3\}$ we can write:

$$\begin{aligned}\Delta &= a_1(a_1 \setminus a_i)((a_1 \setminus a_i) \setminus (a_1 \setminus a_j)) \\ &= a_i(a_i \setminus a_1)((a_1 \setminus a_i) \setminus (a_1 \setminus a_j))\end{aligned}$$

and so:

$$a_i \setminus a_j = a_i^{-1}\Delta = (a_i \setminus a_1)((a_1 \setminus a_i) \setminus (a_1 \setminus a_j))$$

which gives:

$$\begin{aligned}(a_i \setminus a_1) \setminus (a_i \setminus a_j) &= (a_i \setminus a_1) \setminus ((a_i \setminus a_1)((a_1 \setminus a_i) \setminus (a_1 \setminus a_j))) \\ &= (a_1 \setminus a_i) \setminus (a_1 \setminus a_j) \\ (a_i \setminus a_j) \setminus (a_i \setminus a_1) &= ((a_i \setminus a_1)((a_1 \setminus a_i) \setminus (a_1 \setminus a_j))) \setminus (a_i \setminus a_1) \\ &= 1\end{aligned}$$

that is, the cube condition is satisfied for all such triples.

4. There is some cycle, C, containing a_1 and a_2, from (4.2) we know there exists some $b \in A_k$ such that $\Delta = e(C)b$, that is $\Delta = a_1(a_1 \setminus a_2)b = a_2(a_2 \setminus a_1)b$. So again putting $\{i, j\} = \{1, 2\}$ we have
$$a_i \setminus a_3 = a_i^{-1}\Delta = (a_i \setminus a_j)b$$

giving:

$$\begin{aligned}(a_i \setminus a_j) \setminus (a_i \setminus a_3) &= (a_i \setminus a_j) \setminus ((a_i \setminus a_j)b) \\ &= b \\ (a_i \setminus a_3) \setminus (a_i \setminus a_j) &= ((a_i \setminus a_j)b) \setminus (a_i \setminus a_j) \\ &= 1 \\ (a_3 \setminus a_i) \setminus (a_3 \setminus a_j) &= (a_3^{-1}\Delta) \setminus (a_3^{-1}\Delta) \\ &= 1\end{aligned}$$

which shows that the cube condition holds for all such triples.

5. Put $\{i, j, k\} = \{1, 2, 3\}$. The pairs (a_i, a_j) and (a_i, a_k) are cycle-wise disjoint, so $a_i \setminus a_j = a_i \setminus a_k = a_i^{-1}\Delta$ and therefore:

$$(a_i \setminus a_j) \setminus (a_i \setminus a_k) = 1$$

and the cube condition is satisfied.

So the cube condition holds for all triples of atoms, and $(\tilde{C_2}(k), \Delta)$ is a Garside pair.

■

4.2.5 Garside quotients of $\tilde{C}$

The affine Artin group $\tilde{C_2}$ has presentation:

$$\left\langle a, b, c \mid \begin{array}{l} abab = baba; \\ bcbc = cbcb; \\ ac = ca \end{array} \right\rangle$$

In this subsection we describe all Bessis-Garside quotients of $\tilde{C_2}$ preserving the cycles equivalent to these relations. As our notation suggests we will show that the only such quotients are precisely $\tilde{C_2}(k)$, $k = 1, 2, \ldots$.

Theorem 4.2.7 *Let G be a Bessis-Garside group of rank 3 with standard triple (a, b, c) such that $a, b, c, b^{-1}ab, b^{-1}a^{-1}bab, c^{-1}bc, c^{-1}b^{-1}cbc$ are distinct atoms of G satisfying cycles:*

$$\begin{aligned} K(a, b, b^{-1}ab, b^{-1}a^{-1}bab) \\ K(b, c, c^{-1}bc, c^{-1}b^{-1}cbc) \\ K(a, c) \end{aligned}$$

Then G is equal to $\tilde{C_2(k)}$ for some $k = 1, 2, \ldots$, or G is $\tilde{C_2}$.

Proof. We first relabel the atoms of G from the statement, putting $c_0^1 := a$, $a_0 = b$, $c_1^0 := c$, $c_0^0 = b^{-1}ab$, $b_0 = b^{-1}a^{-1}bab$, $b_1 = c^{-1}bc$, and $c_1^1 = c^{-1}b^{-1}cbc$. We put $A_0 := \{a_0, b_0, b_1, c_0^0, c_1^0, c_0^1, c_1^1\}$. So we have:

$$\begin{aligned} {}_{c_1^0}C = K(a_0, c_0^0, b_0, c_0^1) \\ C_{c_0^1} = K(a_0, c_1^0, b_1, c_1^1) \end{aligned}$$

We will proceed by considering triples of atoms from A_0, and in particular the cube condition for such triples, which must hold since G is assumed to be Bessis-Garside. This will allow us to find atoms of G and cycles that must hold in G.

1. The cube condition for the triple $(a_0, c_1^0, c_0^1) \in A_0^3$, implies that $b_1 \setminus c_0^1 = c_1^0 \setminus c_0^0$ is an atom, we call it c_2^0, and confirm that it cannot be equivalent to any of the existing elements of A_0:

$$
\begin{aligned}
c_2^0 = a_0 &\Rightarrow b_1 \setminus a_0 = a_0, \text{but } b_1 \setminus a_0 = c_1^1 \\
c_2^0 = b_0 &\Rightarrow c_0^1 \setminus b_0 = b_1, \text{but } c_0^1 \setminus b_0 = a_0 \\
c_2^0 = b_1 &\Rightarrow K(c_0^0, c_1^0, b_1 \ldots) = K(a_0, c_1^0, b_1, c_1^1) \\
c_2^0 = c_0^0 &\Rightarrow c_0^1 \setminus c_0^0 = b_1, \text{but } c_0^1 \setminus c_0^0 = a_0 \\
c_2^0 = c_1^0 &\Rightarrow c_0^1 \setminus c_1^0 = b_1, \text{but } c_0^1 \setminus c_1^0 = c_1^0 \\
c_2^0 = c_0^1 &\Rightarrow c_0^0 \setminus c_0^1 = c_1^0, \text{but } c_0^0 \setminus c_0^1 = b_0 \\
c_2^0 = c_1^1 &\Rightarrow c_1^0 \setminus c_1^1 = c_1^1, \text{but } c_1^0 \setminus c_1^1 = b_1
\end{aligned}
$$

So we put $A_1 := A_0 \cup \{c_2^0\}$, and note down the following cycles which necessarily hold in G:

$$
\begin{aligned}
C_{a_0} &= K(\ldots, c_0^0, c_1^0, c_2^0, \ldots) \\
C_{c_1^0} &= K(\ldots, c_0^1, b_1, c_2^0, \ldots) \\
C_{c_0^1} = {}_{c_2^0}C &= K(a_0, c_1^0, b_1, c_1^1) \\
{}_{b_1}C &= K(c_0^1, c_1^0) \\
C_{b_1} &= K(c_1^1, c_2^0)
\end{aligned}
$$

2. From the cube condition on (c_0^1, c_1^1, c_1^0) we have $(c_1^1 \setminus c_0^1) \setminus a_0 = a_0$. We call this atom $c_2^1 := c_1^1 \setminus c_0^1$ and confirm that $c_2^1 \notin A_1$:

$$
\begin{aligned}
c_2^1 = a_0 &\Rightarrow c_0^1 \setminus a_0 = c_1^1, \text{but } c_0^1 \setminus a_0 = a_0 \\
c_2^1 = b_0 &\Rightarrow c_0^1 \setminus b_0 = c_1^1, \text{but } c_0^1 \setminus b_0 = a_0 \\
c_2^1 = b_1 &\Rightarrow c_1^1 \setminus b_1 = b_1, \text{but } c_1^1 \setminus b_1 = a_0 \\
c_2^1 = c_0^0 &\Rightarrow c_0^1 \setminus c_0^0 = c_1^1, \text{but } c_0^1 \setminus c_0^0 = a_0 \\
c_2^1 = c_1^0 &\Rightarrow c_1^1 \setminus c_1^0 = c_1^0, \text{but } c_1^1 \setminus c_1^0 = a_0 \\
c_2^1 = c_2^0 &\Rightarrow K(c_0^1, c_1^1, c_2^0, \ldots) = K(c_0^1, b_1, c_2^0, \ldots) \\
c_2^1 = c_0^1 &\Rightarrow a_0 \setminus c_0^1 = c_2^0, \text{but } a_0 \setminus c_0^1 = c_0^0 \\
c_2^1 = c_1^1 &\Rightarrow a_0 \setminus c_1^1 = c_2^0, \text{but } a_0 \setminus c_1^1 = c_1^0
\end{aligned}
$$

So we put $A_2 := A_1 \cup \{c_2^1\}$, and note down the implied cycles:

$$
\begin{aligned}
{}_{a_0}C &= K(\ldots, c_0^1, c_1^1, c_2^1, \ldots) \\
C_{c_1^1} &= K(\ldots, c_2^1, a_0, c_2^0, \ldots)
\end{aligned}
$$

3. From the cube condition on (c_0^0, c_1^0, c_0^1) we see that $c_2^0 \setminus b_1 = c_1^0 \setminus b_0$, is an atom in G, we call this atom a_1, and check that it cannot be an existing element of A_2:

$$
\begin{aligned}
a_1 = a_0 &\Rightarrow b_1 \setminus a_0 = c_2^0, \text{but } b_1 \setminus a_0 = c_1^1 \\
a_1 = b_0 &\Rightarrow c_0^1 \setminus b_0 = b_1, \text{but } c_0^1 \setminus b_0 = a_0 \\
a_1 = b_1 &\Rightarrow K(c_0^1, b_1, c_2^0, b_1) = K(c_0^1, b_1), \text{so } c_0^1 = c_2^0 \\
a_1 = c_0^0 &\Rightarrow b_0 \setminus c_0^0 = c_1^0, \text{but } b_0 \setminus c_0^0 = c_0^1 \\
a_1 = c_1^0 &\Rightarrow b_1 \setminus c_1^0 = c_2^0, \text{but } b_1 \setminus c_1^0 = c_1^1 \\
a_1 = c_2^0 &\Rightarrow K(c_0^0, c_1^0, c_2^0, \ldots) = K(b_0, c_1^0, c_2^0, \ldots) \\
a_1 = c_0^1 &\Rightarrow b_0 \setminus c_0^1 = c_1^0, \text{but } b_0 \setminus c_0^1 = c_0^1 \\
a_1 = c_1^1 &\Rightarrow b_1 \setminus c_1^1 = c_2^0, \text{but } b_1 \setminus c_1^1 = c_1^1 \\
a_1 = c_2^1 &\Rightarrow c_0^1 \setminus c_2^1 = b_1, \text{but } c_0^1 \setminus c_2^1 = c_1^1
\end{aligned}
$$

So we put $A_3 = A_2 \cup \{a_1\}$ and note down the implied cycles:

$$
\begin{aligned}
C_{c_0^0} =_{c_0^1} C &= K(\ldots, b_0, c_1^0, a_1, \ldots) \\
C_{c_1^0} =_{c_1^1} C &= K(a_1, c_0^1, b_1, c_2^0) \\
C_{b_0} =_{b_1} C &= K(c_0^1, c_1^0) \\
C_{a_0} =_{a_1} C &= K(\ldots, c_0^0, c_1^0, c_2^0, \ldots)
\end{aligned}
$$

4. The cube condition on (b_1, c_2^0, c_1^1, b_1) implies that $c_1^1 \setminus a_1 = c_2^0 \setminus a_0$, we denote this atom by b_2 and check that $b_2 \notin A_3$:

$$
\begin{aligned}
b_2 = a_0 &\Rightarrow K(c_2^1, a_0, c_2^0, a_0, \ldots) = K(c_2^1, a_0), \text{so } c_2^0 = c_2^1 \\
b_2 = a_1 &\Rightarrow K(c_2^1, a_0, c_2^0, a_1, \ldots) = K(c_0^1, b_1, c_2^0, a_1) \\
b_2 = b_0 &\Rightarrow a_0 \setminus b_0 = c_2^0, \text{but } a_0 \setminus b_0 = c_0^0 \\
b_2 = b_1 &\Rightarrow a_0 \setminus b_1 = c_2^0, \text{but } a_0 \setminus b_1 = c_1^0 \\
b_2 = c_0^0 &\Rightarrow a_0 \setminus c_0^0 = c_2^0, \text{but } a_0 \setminus c_0^0 = c_0^0 \\
b_2 = c_1^0 &\Rightarrow a_0 \setminus c_1^0 = c_2^0, \text{but } a_0 \setminus c_1^0 = c_1^0 \\
b_2 = c_2^0 &\Rightarrow K(c_2^1, a_0, c_2^0, c_2^0, \ldots), \text{so } a_0 = c_2^0 \\
b_2 = c_0^1 &\Rightarrow a_0 \setminus c_0^1 = c_2^0, \text{but } a_0 \setminus c_0^1 = c_0^0 \\
b_2 = c_1^1 &\Rightarrow a_0 \setminus c_1^1 = c_2^0, \text{but } a_0 \setminus c_1^1 = c_1^0 \\
b_2 = c_2^1 &\Rightarrow K(a_1, c_1^1, c_2^1, \ldots) = K(c_0^1, c_1^1, c_2^1, \ldots)
\end{aligned}
$$

So we put $A_4 = A_3 \cup \{b_2\}$ and note down the implied cycles:

$$
\begin{aligned}
C_{c_1^1} &= K(a_0, c_2^0, b_2, c_2^1) \\
C_{c_2^0} =_{c_2^1} C &= K(\ldots, a_1, c_1^1, b_2, \ldots) \\
C_{b_1} =_{b_2} C &= K(c_1^1, c_2^0) \\
C_{a_1} =_{a_0} C &= K(\ldots, c_0^1, c_1^1, c_2^1, \ldots)
\end{aligned}
$$

To summarise we have, for $i = 2$, a subset of the atoms of G, $B_i = \{a_m, b_l, c_l^m \mid m \in \{0, 1\}; l = 0, 1, \ldots i\}$, and a set of cycles which hold in G:

$$
\begin{aligned}
C_{a_m} =_{a_{|m-1|}} C &= K(c_0^m, c_1^m, \ldots, c_i^m, \ldots), \text{for } m \in \{0, 1\} \\
C_{b_l} =_{b_{l+1}} C &= K(c_l^1, c_{l+1}^0), \text{for } l = 0, 1, \ldots, i-1 \\
C_{c_l^0} =_{c_l^1} C &= K(a_1, c_l^1, b_{l+1}, c_{l+2}^0), \text{for } l = 1, 2, \ldots, i-1 \\
C_{c_l^1} =_{c_{l+2}^0} C &= K(a_0, c_{l+1}^0, b_{l+1}, c_{l+1}^1), \text{for } l = 0, 1, \ldots, i-2 \\
C_{c_0^0} =_{c_0^1} C &= K(\ldots, b_0, c_1^0, a_1, \ldots) \\
C_{c_i^0} =_{c_i^1} C &= K(\ldots, a_1, c_{i-1}^1, b_i, \ldots) \\
C_{c_{i-1}^1} &= K(a_0, c_i^0, b_i, c_i^1) \\
& K(a_0, c_0^0, b_0, c_0^1)
\end{aligned}
$$

We proceed by induction on i. Assume we have B_j and associated cycles for some $j \geq 2$

Consider the triple $(c^0_{j-1}, c^0_j, c^1_{j-2})$, the cube condition holding for this triple implies that: $(c^0_j \setminus c^0_{j-1}) \setminus a_1 = a_1$; we call $c^0_{j+1} := c^0_j \setminus c^0_{j-1}$ and determine whether it is possible that $c^0_{j+1} \in B_j$:

$$\begin{aligned}
c^0_{j+1} = a_0 &\Rightarrow c^0_j \setminus a_0 = a_0, \text{but } c^0_j \setminus a_0 = b_j \\
c^0_{j+1} = a_1 &\Rightarrow K(a_1, a_1, c^1_{j-1}, b_j, \ldots), \text{so } a_1 = c^1_{j-1} \\
c^0_{j+1} = b_l &\Rightarrow c^0_l \setminus b_l = c^0_{l+1}, \text{but } c^0_l \setminus b_l = b_l; \text{for} l = 0, 1, \ldots j-1 \\
c^0_{j+1} = b_j &\Rightarrow K(c^0_0, \ldots, c^0_{j-1}, c^0_j, b_j) = K(a_0, c^0_j, b_0, c^1_j), \text{so } a_0 = c^0_{j-1} \\
c^0_{j+1} = c^0_l &\Rightarrow K(c^0_0, \ldots, c^0_{l-1}, c^0_l, \ldots, c^0_j, c^0_l, \ldots), \text{so } c^0_j = c^0_{l-1}; \text{for } l = 1, 2, \ldots, j \\
c^0_{j+1} = c^1_l &\Rightarrow c^1_{j-1} \setminus c^1_l = b_j, \text{but } c^1_{j-1} \setminus c^1_l = c^1_j; \text{for } l = 0, 1, \ldots, j-2, \widehat{j-1}, j \\
c^0_{j+1} = c^1_{j-1} &\Rightarrow K(c^1_{j-1}, a_1, c^1_{j-1}, b_j), \text{so } a_1 = b_j
\end{aligned}$$

and we can conclude that the only possibility is $c^0_j \setminus c^0_{j-1} = c^0_0$.

If we assume that this is not the case, and put $\bar{A}_1 = B_j \cup \{c^0_{j+1}\}$, then we have the following cycles:

$$\begin{aligned}
C_{a_0} =_{a_1} C &= K(\ldots, c^0_0, c^0_1, \ldots, c^0_j, c^0_{j+1}, \ldots) \\
C_{c^0_j} =_{c^1_j} C &= K(c^0_{j+1}, a_1, c^1_{j-1}, b_j) \\
C_{c^1_{j-1}} =_{c^0_{j+1}} C &= K(a_0, c^0_j, b_j, c^1_j) \\
C_{b_j} &= K(c^0_{j+1}, c^1_j)
\end{aligned}$$

From this position we can show that we necessarily find ourselves in the situation of the induction hypothesis with j replaced by $j+1$. As before we do this by considering the cube condition on various triples, and applying the fact that G is assumed to be Garside.

1. From the triple (c^0_{j+1}, c^1_j, b_j) we find $c^1_j \setminus a_1 = c^0_{j+1} \setminus a_0$; we call this atom b_{j+1} and confirm that $b_{j+1} \notin \bar{A}_1$:

$$\begin{aligned}
b_{j+1} = a_0 &\Rightarrow K(a_1, c^1_j, a_0, \ldots) = K(a_0, c^0_j, b_j, c^1_j), \text{so } a_1 = b_j \\
b_{j+1} = a_1 &\Rightarrow K(a_0, c^0_{j+1}, a_1, \ldots) = K(a_1, c^1_{j-1}, b_j, c^0_{j+1}), \text{so } a_0 = b_j \\
b_{j+1} = b_l &\Rightarrow a_0 \setminus b_l = c^0_{j+1}, \text{but } a_0 \setminus b_l = c^0_l; \text{for } l = 0, 1, \ldots, j \\
b_{j+1} = c^0_l &\Rightarrow a_0 \setminus c^0_l = c^0_{j+1}, \text{but } a_0 \setminus c^0_l = c^0_l; \text{for } l = 0, 1, \ldots, j \\
b_{j+1} = c^0_{j+1} &\Rightarrow K(a_0, c^0_{j+1}, c^0_{j+1}, \ldots), \text{so } a_0 = c^0_{j+1} \\
b_{j+1} = c^1_l &\Rightarrow a_1 \setminus c^1_l = c^1_j, \text{but } a_1 \setminus c^1_l = c^1_l; \text{for } l = 0, 1, \ldots, j-1 \\
b_{j+1} = c^1_j &\Rightarrow K(a_1, c^1_j, c^1_j, \ldots), \text{so } a_1 = c^1_j.
\end{aligned}$$

so we put $\bar{A}_2 = \bar{A}_1 \cup \{b_{j+1}\}$ and make note of the cycles:

$$C_{b_j} =_{b_{j+1}} C = K(c^0_{j+1}, c^1_j), C_{c^1_j} \quad = K(\dots, a_0, c^0_{j+1}, b_{j+1}, \dots) C_{c^0_{j+1}} = K(\dots, a_1, c^1_j, b_{j+1}, \dots)$$

2. From the triple (a_1, c^1_j, b_j) we find $b_{j+1} \setminus c^0_{j+1} = c^1_j \setminus c^1_{j-1}$, we call this atom c^1_{j+1} and check that $c^1_{j+1} \notin \bar{A}_2$:

$$\begin{aligned}
c^1_{j+1} = a_0 &\Rightarrow c^1_0 \setminus a_0 = c^1_1, \text{but } c^1_0 \setminus a_0 = a_0 \\
c^1_{j+1} = a_1 &\Rightarrow c^1_0 \setminus a_1 = c^1_1, \text{but } c^1_0 \setminus a_1 = b_1 \\
c^1_{j+1} = b_l &\Rightarrow c^1_l \setminus b_l = c^1_{l+1}, \text{but } c^1_l \setminus b_l = a_0; \text{for } l = 0, 1, \dots j-1 \\
c^1_{j+1} = b_j &\Rightarrow c^1_j \setminus b_j = b_j, \text{but } c^1_j \setminus b_j = a_0 \\
c^1_{j+1} = b_{j+1} &\Rightarrow K(a_0, c^0_{j+1}, b_{j+1}, b_{j+1}, \dots), \text{so } c^0_{j+1} = b_{j+1} \\
c^1_{j+1} = c^0_l &\Rightarrow a_0 \setminus c^0_l = c^0_{j+1}, \text{but } a_0 \setminus c^0_l = c^0_l; \text{for } l = 0, 1, \dots j \\
c^1_{j+1} = c^0_{j+1} &\Rightarrow K(a_0, c^0_{j+1}, b_{j+1}, c^0_{j+1}, \dots), \text{so } a_0 = b_{j+1} \\
c^1_{j+1} = c^1_l &\Rightarrow K(c^1_0, \dots, c^1_{l-1}, c^1_l, \dots, c^1_j, c^1_l, \dots), \text{so } c^1_j = c^1_{l-1}; \text{for } l = 1, 2, \dots, j
\end{aligned}$$

so we put $\bar{A}_3 = \bar{A}_2 \cup \{c^1_{j+1}\}$ and make note of the implied cycles:

$$\begin{aligned}
C_{a_1} =_{a_0} C &= K(\dots, c^1_0, c^1_1, \dots, c^1_j, c^1_{j+1}, \dots), \\
C_{c^1_j} &= K(a_0, c^0_{j+1}, b_{j+1}, c^1_{j+1}), \\
C_{c^0_{j+1}} =_{c^1_{j+1}} C &= K(\dots, a_1, c^1_j, b_{j+1}, \dots)
\end{aligned}$$

Thus we have a set of atoms B_{j+1} with associated cycles as described above.

So by this induction we have (for $i = 0$) a set of atoms $C_i = \{a_m, b_l, c^m_l \mid m \in \{0, 1\},\ l = i, i+1, \dots\}$; and cycles:

$$\begin{aligned}
C_{a_m} =_{a_{|m-1|}} C &= K(c^m_i, c^m_{i+1}, \dots), \text{for } m \in \{0, 1\} \\
C_{b_l} =_{b_{l+1}} C &= K(c^1_l, c^0_{l+1}), \text{for } l = i, i+1, \dots \\
{}_{c^0_{i+1}} C &= K(a_0, c^0_i, b_i, c^1_i) \\
C_{c^1_l} =_{c^0_{l+2}} C &= K(a_0, c^0_{l+1}, b_{l+1}, c^1_{l+1}), \text{for } l = i, i+1, \dots \\
C_{c^0_i} =_{c^1_i} C &= K(\dots, b_i, c^0_{i+1}, a_1, \dots) \\
C_{c^0_{l+1}} =_{c^1_{l+1}} C &= K(a_1, c^1_l, b_{l+1}, c^0_{l+2}), \text{for } l = i, i+1, \dots
\end{aligned}$$

Assume we have C_j and associated cycles for some $j \leq 0$. We can show that we necessarily find ourselves in the situation of this induction hypothesis with j replaced by $j-1$.

1. From (a_1, c_{j+1}, b_{j+1}) we find $c_j^1 \backslash (a_1 \backslash c_{j+1}^0) = c_{j+1}^1$. We call this atom $c_{j-1}^1 := a_1 \backslash c_{j+1}^0$, and confirm that c_{j-1}^1 is not in C_j:

$$\begin{aligned}
c_{j-1}^1 = a_0 &\Rightarrow a_0 \setminus c_j^1 = c_j^1, \text{but } a_0 \setminus c_j^1 = c_j^0\\
c_{j-1}^1 = a_1 &\Rightarrow K(\ldots, b_j, c_{j+1}^0, a_1, a_1, \ldots), \text{so } c_{j+1}^0 = a_1\\
c_{j-1}^1 = b_j &\Rightarrow c_j^1 \setminus b_j = c_{j+1}^1, \text{but } c_j^1 \setminus b_j = a_0\\
c_{j-1}^1 = b_l &\Rightarrow a_1 \setminus b_l = b_l, \text{but } a_1 \setminus b_l = c_{l-1}^1; \text{for } l = j+1, j+2, \ldots\\
c_{j-1}^1 = c_l^0 &\Rightarrow c_l^0 \setminus c_l^1 = c_j^1, \text{but } c_l^0 \setminus c_l^1 = b_l; \text{for } l = j, j+1, \ldots\\
c_{j-1}^1 = c_l^0 &\Rightarrow K(\ldots, c_l^1, c_j^1, \ldots, c_l^1, c_{1+1}^1, \ldots), \text{so } c_j^1 = c_{l+1}^1; \text{for } l = j, j+1, \ldots
\end{aligned}$$

So we put $C_j' = C_j \cup \{c_{j-1}^1\}$, and note that we must have cycles:

$$\begin{aligned}
C_{a_1} =_{a_0} C &= K(\ldots, c_{j-1}^1, c_j^1, c_{j+1}^1, \ldots)\\
C_{c_j^0} =_{c_j^1} C &= K(a_1, c_{j-1}^1, b_j, c_{j+1}^0)\\
C_{c_{j-1}^1} =_{c_{j+1}^0} C &= K(a_0, c_j^0, b_j, c_j^1)\\
{}_{b_j}C &= K(c_j^0, c_{j-1}^1)
\end{aligned}$$

2. We can factorise $\Delta = c_{j-1}^1 a_0 c_j^0$, so we have cycle containing $K(\ldots, c_{j-1}^1, a_0, \ldots)$

3. From (a_0, b_j, c_{j-1}^1) we have $c_j^0 \backslash (a_0 \backslash c_{j-1}^1) = c_{j+1}^0$. We call this atom $c_{j-1}^0 := a_0 \backslash c_{j-1}^1$, and confirm that c_{j-1}^0 is not in C_j':

$$\begin{aligned}
c_{j-1}^0 = a_0 &\Rightarrow K(\ldots, c_{j-1}^1, a_0, a_0, \ldots), \text{so } c_{j-1}^1 = a_0\\
c_{j-1}^0 = a_1 &\Rightarrow c_{j-1}^1 \setminus a_1 = a_0, \text{but } c_{j-1}^1 \setminus a_1 = b_j\\
c_{j-1}^0 = b_1 &\Rightarrow c_l^0 \setminus b_l = c_{l+1}^0, \text{but } c_l^0 \setminus b_l = b_l; \text{for } l = j, j+1, \ldots\\
c_{j-1}^0 = c_1^0 &\Rightarrow K(\ldots, c_l^0, c_j^0, \ldots, c_l^0, c_{1+1}^0, \ldots), \text{so } c_j^0 = c_{l+1}^0; \text{for } l = j, j+1, \ldots\\
c_{j-1}^0 = c_{j-1}^1 &\Rightarrow c_{j-1}^1 \setminus c_{j+1}^1 = c_j^0, \text{but } c_{j-1}^1 \setminus c_{j+1}^0 = b_j\\
c_{j-1}^0 = c_l^1 &\Rightarrow c_l^0 \setminus c_l^1 = c_{l+1}^0, \text{but } c_l^0 \setminus c_l^1 = b_l; \text{for } l = j, j+1, \ldots
\end{aligned}$$

So we put $C_j'' = C_j' \cup \{c_{j-1}^0\}$, and note that we must have cycles:

$$\begin{aligned}
C_{a_0} =_{a_1} C &= K(\ldots, c_{j-1}^0, c_j^0, c_{j+1}^0, \ldots)\\
{}_{c_j^0}C &= K(\ldots, c_{j-1}^1, a_0, c_{j-1}^0, \ldots)
\end{aligned}$$

4. From (a_0, c^0_{j-1}, c^0_j) we have $c^0_j \setminus (c^0_{j-1} \setminus a_0) = a_1$. We call this atom $b_{j-1} := c^0_{j-1} \setminus a_0$, and confirm that b_{j-1} is not in C''_j:

$$\begin{aligned}
b_{j-1} = a_0 &\Rightarrow K(\dots, c^1_{j-1}, a_0, c^0_{j-1}, a_0 \dots), \text{so } c^1_{j-1} = c^0_{j-1} \\
b_{j-1} = a_1 &\Rightarrow c^1_{j-1} \setminus a_1 = a_0, \text{but } c^1_{j-1} \setminus a_1 = b_j \\
b_{j-1} = b_l &\Rightarrow a_0 \setminus b_l = c^0_{j-1}, \text{but } a_0 \setminus b_l = c^0_l; \text{for } l = j, j+1, \dots \\
b_{j-1} = c^0_{j-1} &\Rightarrow K(\dots, c^0_{j-1}, a_0, c^0_{j-1}, c^0_{j-1}, \dots), \text{so } a_0 = c^0_{j-1} \\
b_{j-1} = c^0_l &\Rightarrow a_0 \setminus c^0_l = c^0_{j-1}, \text{but } a_0 \setminus c^0_l = c^0_l; \text{for } l = j, j+1, \dots \\
b_{j-1} = c^1_l &\Rightarrow c^1_l \setminus a_1 = c^0_j, \text{but } c^1_l \setminus a_1 = b_{l+1}; \text{for } l = j, j+1, \dots
\end{aligned}$$

So we put $C_{j-1} = C''_j \cup \{b_{j-1}\}$, and note that we must have cycles:

$$\begin{aligned}
{}_{c^0_j}C &= K(a_0, c^0_{j-1}, b_{j-1}, c^1_{j-1}) \\
C_{c^0_{j-1}} =_{c^1_{j-1}} C &= K(\dots, b_{j-1}, c^0_j, a_1, \dots) \\
C_{b_{j-1}} =_{b_j} C &= K(c^1_{j-1}, c^0_j)
\end{aligned}$$

So if we have none of the possible quotients $c^0_j \setminus c^0_{j-1} = c^0_0$, then we have a group with cycle presentation:

$$\tilde{C}_2(\infty) = \left\langle \begin{array}{lr} a_m, & m \in \{0,1\} \\ b_l, & l \in \mathbb{Z} \\ c^m_l, & l \in \mathbb{Z},\ m \in \{0,1\} \end{array} \middle| \begin{array}{lr} K(\dots, c^m_{-1}, c^m_0, c^m_1, \dots), & m \in \{0,1\} \\ K(c^0_{l+1}, c^1_l), & l \in \mathbb{Z} \\ K(a_m, c^m_l, b_{l+m}, c^{|m-1|}_{l+2m}), & l \in \mathbb{Z}, m \in \{0,1\} \end{array} \right\rangle$$

Now by [21], the affine Artin group $\tilde{C}_2$, with generators c^1_0, a_0, c^0_1 and $\Delta = c^1_0 a_0 c^0_1$, meets all the assumptions of the statement, and it is clear that none of the possible relations $c^0_j \setminus c^0_{j-1} = c^0_0$ hold in $\tilde{C}_2$, so we may conclude that $\tilde{C}_2(\infty) = \tilde{C}_2$.

Alternatively if we take $c^0_j \setminus c^0_{j-1} = c^0_0$, then we can show that $G = \tilde{C}_2(j-1)$ as follows:

1. From the triple $(c^0_{j-1}, c^0_j, c^1_{j-2})$ discussed above we now choose $c^0_j \setminus c^0_{j-1} = c^0_0$, which gives us the cycles:

2. (a_1, c^0_0, c^0_j) gives us cycles:

$$\begin{aligned}
C_{c^0_0} =_{c^1_0} C &= K(a_1, c^1_j), b_0, c^0_1) \\
C_{b_j} =_{b_0} C &= K(c^0_0, c^1_j)
\end{aligned}$$

which shows that $\frac{\tilde{C}_2}{c^0_{j+1} = c^0_0}$ is $\tilde{C}_2(j-1)$. ■

4.3 Conditions for Bessis-Garside groups

In subsections 4.2.2 and 4.2.4 we saw a set of properties of the groups $\tilde{C_2}(k)$ which were sufficient to demonstrate that these groups were Bessis-Garside. In this subsection we generalise those properties in order to give a set of criteria on cycle presented groups that are necessary and sufficient for such a group to admit a Bessis-Garside structure.

Theorem 4.3.1 *Let G be a cycle presented group generated by set of atoms A and with relations given by a set of cycles C. Suppose that:*

Condition 4.3.2 *For each $a \in A$ the right adjacency set of a, $R(a)$, is the underlying set of some cycle C_a; and $\Delta = ae(C_a)$ is independent of the choice of $a \in A$*

holds, then (G, Δ) is a Bessis-Garside pair of rank 3. Conversely any Bessis-Garside group of rank 3, with regular standard triple, admits a cycle presentation satisfying Condition 4.3.2.

Proof. We saw in the proof of Lemma 4.2.6 that Condition 4.3.2 is sufficient to show that the cube condition is satisfied for all triples of atoms in A^3. If G is a Bessis group, then this is enough to show that G is in fact Bessis-Garside.

In the proof of Lemma 4.2.3 we saw that if G satisfies:

Condition 4.3.3 *G has a triple of atoms $(a, b, c) \in A^3$ satisfying $p_1((a, b, c).B_3) = A$, and for every cycle $K \in C$ there exists an atom $x \in A$ such that $xe(K) = abc$*

then G is Bessis.

So to prove the first implication of the theorem it is sufficient to show that Condition 4.3.3 follows from Condition 4.3.2.

We choose some $a \in A$, and $b, c \in R(a)$ such that $b \setminus c = c$, so that $\Delta = abc$. Then for any $x \in A$ we have $xe(C_x) = xyz = \Delta$, for some $y, z = y \setminus z \in R(x)$, and so there is a sequence of triples:

$$(a, b, c) = (t_0, u_0, v_0) \dots (t_k, u_k, v_k) = (x, y, z)$$

with:

$$\begin{aligned}
(t_{i+1}, u_{i+1}, v_{i+1}) &= ((u_i^{-1}, t_i^{-1}; j)(u_i^{-1}, t_i^{-1}; j+1)^{-1}, (u_i^{-1}, t_i^{-1}; j+1)(u_i^{-1}, t_i^{-1}; j+2)^{-1}, v_i) \\
&= (t_i, u_i, v_i).\sigma_1^j; \text{for } j \in \mathbb{Z}, \text{or} \\
(t_{i+1}, u_{i+1}, v_{i+1}) &= (t_i, (v_i^{-1}, u_i^{-1}; j)(v_i^{-1}, v_i^{-1}; j+1)^{-1}, (v_i^{-1}, u_i^{-1}; j+1)(v_i^{-1}, u_i^{-1}; j+2)^{-1}) \\
&= (t_i, u_i, v_i).\sigma_2^j; \text{for } j \in \mathbb{Z}
\end{aligned}$$

since $abc = xyz$, and all relations in G come from cycles.
So $p_1((a, b, c).B_3) = A$, that is we may take (a, b, c) as the triple required by Condition 4.3.3.

Suppose $K(\ldots,x,y,\ldots)$ is a cycle in G. Then we have $C_x = K(\ldots,v,y,z,\ldots)$ for some $v,z \in A$, and we have a cycle $K(\ldots,u,x,v,\ldots)$ for some $u \in A$, since $v \in R(x)$. Then

$$\Delta = xe(C_x) = xyz = xvy = uxy = ue(K(\ldots,x,y,\ldots))$$

and so Condition 4.3.3 is a consequence of Condition 4.3.2, as required.

It remains to show the converse.

Suppose G is a Bessis-Garside group of rank 3. For an arbitrary atom $x = x_1.g \in A = \{x_1.g \mid g \in B_3\}$ we consider $R(x)$.

Suppose $|R(x)| = 2$. Since $x = x_1.g$, $\Delta = x_1x_2x_3 = x_1.gx_2.gx_3.g$. By lemma 4.1.2 we have $x_2.g, x_3.g \in R(x)$, and by Proposition 4.1.4, $x_2.g \neq x_3.g$; so $R(x) = \{x_2.g, x_3.g\}$. Again using lemma 4.1.2, $x_3.g = x_2.g \setminus x_3.g$. We have $\Delta = xx_3.g(x_3.g \setminus x_2.g)$, so $x_3.g \setminus x_2.g \in R(x)$, and therefore $x_3.g \setminus x_2.g = x_2.g$, and the cycle $K(x_2.g, x_3.g)$ holds in G, and $\Delta = xe(K(x_2.g, x_3.g))$, satisfying condition 4.3.2.

Suppose $|R(x)| \geq 3$, and $R(x) = \{y_i\}_{i \in I}$. For all y_i, $i \in I$, we have $xy_i \preceq \Delta$ (because $x \preceq \Delta$, $y_i \preceq \Delta$, and $x \vee y_i = xy_i$). So for all $i \in I$, there exists $z_i \in A$ such that $\Delta = xy_iz_i$, and we have $x^{-1}\Delta = y_iz_i$ for all i. So $y_i \setminus y_j = z_i \in R(x)$ for distinct $i,j \in I$. Choosing some $i_0 \in I$ we put $w_0 := y_{i_0}$, $w_1 = w_0 \setminus y_{i_0}$, and $w_{k+1} = w_k \setminus y_{i_0}$, so that $R(x)$ is the underlying set of the cycle $C_x = K(\ldots,w_0,w_1,w_2,\ldots)$, and clearly $xe(C_x) = \Delta$, and Condition 4.3.2 is again satisfied.
This completes the proof.

■

Proposition 4.3.4 *Let G be a cycle presented group satisfying Condition 4.3.2,with set of atoms A, and set of cycles C. Then $|A| = |C|$*

Proof. As Condition 4.3.2 implies G is Garside, we may define a map $\kappa : C \to A$ by $\kappa(C) = \Delta(e(C))^{-1}$. For $a \in A$, $\kappa(C_a) = a$ so that κ is surjective. If $\kappa(C) = \kappa(D) = a$, then choosing $c_1, c_2 = c_1 \setminus c_2 \in C$, and $d_1, d_2 = d_1 \setminus d_2 \in D$ we have $c_1, c_2, d_1, d_2 \in R(a)$ so that $C = D = C_a$ and κ is injective. ■

We can translate this Theorem 4.3.1 into the language of cycle graphs from Section 3.3:

Corollary 4.3.5 *Let Γ be a directed graph, with vertices V, and edges E. The* ***right adjacent set*** *of a vertex v is defined to be $R(v) := \{w \in V \mid (v,w) \in E\}$, and we call C_v the subgraph induced by $R(v)$. Suppose that for all $v \in V$, we have $C_a = \Gamma(F_2)$ or $C_a = \Gamma(I_2(k))$ for some $k \geq 2$, for the cycle presentations of F_2 and $I_2(k)$ given in the Section 3.3, and:*

Condition 4.3.6 *For any two vertices $u, v \in V$ there exists a finite sequence of 4-tuples of vertices $(p_i, q_i, r_i, s_i), i = 0, 1, \ldots, k$; with $p_0 = u$, $p_k = v$; such that $p_i, q_i \in R(s_i), q_i, r_i \in R(p_i), r_i \in R(q_i)$; and either $p_{i+1} = p$, or $r_{i+1} = r_i$ and $s_{i+1} = s_i$ for all $i = 0, 1, \ldots, k$.*

Then there exists Bessis-Garside group G with $\Gamma = \Gamma(G)$

Proof. The assumptions of the theorem simply imply that G satisfies Condition 4.3.2, so that G is Bessis-Garside. ■

Proposition 4.3.7 *Let G be a cycle presented group generated by set of atoms A and with relations given by a set of cycles C. Then Condition 4.3.2 is equivalent to:*

Condition 4.3.8 *For each $a \in A$ the right adjacency set of a, $R(a)$, and the left adjacency set of a, $L(a)$ are the underlying sets of cycles denoted C_a and ${}_aC$ respectively; the cycle graph of G, $\Gamma(G)$ is path-connected.*

Proof. First suppose that Condition 4.3.2 holds. If $b \in R(a)$, then since G is cycle presented we have cycle $K(\ldots, c, a, b, \ldots)$ for some $c \in A$; and since two atoms determine a cycle, we have a $1-1$ correspondence between $R(a)$ and $L(a)$. Suppose $C_a = K(\ldots, b_i, b_{i+1}, \ldots)$ for $i \in \mathbb{Z}$ or $\mathbb{Z}_k$ for some $k \geq 2$; then $\Delta = ab_ib_{i+1} = (ab_ia^{-1})(ab_{i+1}a^{-1})a$, so that $ab_ia^{-1} \in L(a)$ and $(ab_ia^{-1}) \setminus (ab_{i+1}a^{-1}) = ab_{i+1}a^{-1}$ for all $b_i \in R(a)$. That is $L(a)$ is the underlying set of the cycle ${}_aC = K(\ldots, ab_ia^{-1}, ab_{i+1}a^{-1}, \ldots)$. We saw in Corollary 4.3.5 that $ae(C_a) = be(C_b)$ is equivalent to a rewriting by cycle relations, which we can think of in cycle diagrammatic terms as a sequence of form given in Condition 4.3.6. In fact we may further specify that if $r_{i+1} = r_i$ and $s_{i+1} = s_i$ then either: $p_{i+1} = q_i$ and $q_{i+1} = q_i \setminus p_i$; or $q_{i+1} = p_i$ and $p_i \setminus p_{i+1} = q_i$. That is cycle relations moving only one step along the cycle, so that p_i and p_{i+1} are always adjacent in $\Gamma(G)$, and the sequence $p_0, \ldots, p_k$ is a path from a to b, and $\Gamma(G)$ is path-connected as required.

Conversely suppose Condition 4.3.8 holds, and we have a path $u = \bar{p}_0, \ldots, \bar{p}_k = v$ between vertices/atoms u and v. We construct a sequence of 4-tuples as in Condition 4.3.6. Put $p_0 = u$ and choose $q_0, r_0 = q_0 \setminus r_0 \in R(p_0)$. Then we have $C_{p_0} = (\ldots, a, q_0, r_0, \ldots)$, and since $L(q_0)$ is the underlying set of a cycle, we have ${}_{q_0}C = (\ldots, s_0, p_0, a, \ldots)$ for some s_0, so that $p_0, q_0 \in R(s_0)$. So we have initial 4-tuple (p_0, q_0, r_0, s_0) satisfying Condition 4.3.6. Now suppose we have a suitable sequence for $i = 0, 1, \ldots, j$ and $p_j = \bar{p}_l$. If $\bar{p}_{l+1} \in R(s_j)$ then we can write the next 4-tuple in the sequence: $(p_{j+1}, q_{j+1}, r_{j+1}, s_{j+1}) = (\bar{p}_{l+1}, \bar{p}_{l+1} \setminus p_j, r_i, s_i)$. If $\bar{p}_{l+1} \notin R(s_j)$, then if $\bar{p}_{l+1} \in R(p_i)$ we have $C_{p_i} = K(\ldots, \bar{p}_{l+1}, r_{i+1}, \ldots, q_i, r_i, \ldots)$ for some r_{i+1}, and we may write $(p_{j+1}, q_{j+1}, r_{j+1}, s_{j+1}) = (p_j, \bar{p}_{l+1}, r_{i+1}, s_{i+1})$, with s_{i+1} satisfying the conditions of Condition 4.3.6 found in the same way as s_0. ALternatively we may have $\bar{p}_{l+1} \notin R(s_j)$ and $\bar{p}_l \setminus \bar{p}_{l+1} \in R(p_i)$ so that $C_{p_i} = K(\ldots, \bar{p}_l \setminus \bar{p}_{l+1}, r_{i+1}, \ldots, q_i, r_i, \ldots)$ for some r_{i+1}, and we can write $(p_{j+1}, q_{j+1}, r_{j+1}, s_{j+1}) = (p_j, \bar{p}_l \setminus \bar{p}_{l+1}, r_{i+1}, s_{i+1})$, again with s_{i+1} found as above. In either case we find ourselves back in the case with $\bar{p}_{l+1} \in R(s_j)$. So in this manner we can construct a sequence of applications of cycle relations from $ue(C_u)$ to $ve(C_v)$ as required. ■

4.4 $\tilde{A}_2(k)$

We introduce a family of groups $\tilde{A}_2(k)$, $k \geq 1$, with cycle presentations.

Definition 4.4.1 *We denote by $\tilde{A}_2(k)$ the cycle presented group, with set of generators:*

$$A_{\tilde{A}_2(k)} := \{a_m, c_l^m \mid l \in \mathbb{Z}_{k+1}, m \in \{0,1\}\}$$

and relations given by cycles:

$$\begin{aligned} &K(c_0^m, c_1^m, \ldots, c_k^m), && m \in \{0,1\} \\ &K(a_m, c_l^m, c_{l+m}^{|m-1|}), && m \in \{0,1\},\ l \in \mathbb{Z}_{k+1}. \end{aligned}$$

These are in fact the groups given by Bessis and Corran as the case $n = 2$ of theorem 7.1 in [6]. Our generators a_m and c_l^m become the symmetric ($a_{p,q}$) and asymmetric (a_r) generators of their presentation respectively, while the relations of the cycles $K(a_m, c_l^m, c_{l+m}^{|m-1|})$ and $K(c_0^m, c_1^m, \ldots, c_k^m)$ are precisely those of types $\mathcal{R}_5$ and $\mathcal{R}_6$.

We can easily write out the right adjacency sets of the atoms:

$$\begin{aligned} R(a_m) &= \{c_i^m \mid i \in \mathbb{Z}_{k+1}\} \\ R(c_l^m) &= \{a_{|m-1|}, c_{l+1}^m, c_{l+m}^{|m-1|}\} = \{a_{|m-1|}, c_{l+m}^{|m-1|}, c_{(l+m)+(|m-1|)}^m\} \end{aligned}$$

and the left adjacency sets:

$$\begin{aligned} L(a_m) &= \{c_i^{|m-1|} \mid i \in \mathbb{Z}_{k+1}\} \\ L(c_l^m) &= \{a_m, c_{l-1}^m, c_{l-1+m}^{|m-1|}\} = \end{aligned}$$

and conclude that the left and right adjacency sets of all atoms are the underlying sets of cycles.

If we put $\Delta = c_0^1 a_0 c_1^0$, then we have the following factorisations of Δ:

$$\begin{aligned} \Delta &= a_m e(c_0^m, c_1^m, \ldots, c_k^m), && m \in \{0,1\} \\ &= e(c_0^{|m-1|}, c_1^{|m-1|}, \ldots, c_k^{|m-1|}) a_m && m \in \{0,1\} \\ &= c_l^m e(a_{|m-1|}, c_{l+m}^{|m-1|}, c_{(l+m)+(|m-1|)}^m), && m \in \{0,1\},\ l \in \mathbb{Z}_{k+1} \\ &= e(a_m, c_{l-1}^m, c_{l-1+m}^{|m-1|}) c_l^m. && m \in \{0,1\},\ l \in \mathbb{Z}_{k+1} \end{aligned}$$

So from Theorem 4.3.1, we can conclude that the group $\tilde{A}_2(k)$ is Bessis-Garside for each $k = 1, 2, \ldots$.

Theorem 4.4.2 *Let G be a Bessis-Garside group, with standard triple (a,b,c), and a, b, c, $b^{-1}ab$, $c^{-1}bc$, $c^{-1}ac$ distinct atoms satisfying cycles $K(a,b,b^{-1}ab)$, $K(b,c,c^{-1}bc)$, and $K(a,c,c^{-1}ac)$. Then G is equal to $\tilde{A}_2(k)$ for some $k = 1,2,\ldots$ or G is the affine Artin group $\tilde{A}_2$.*

Proof.

We begin by relabelling $a_0 := b$, $a_1 := c^{-1}ac$, $c_0^0 = b^{-1}ab$, $c_1^0 := c$, $c_0^1 := a$, and $c_1^1 = c^{-1}bc$. If we consider the cube condition triples from $\{a_0, c_1^0, c_0^1\}$, which must be satisfied since G is Garside, we find that:

$$c_0^0 \setminus c_1^0 = a_0 \setminus c_1^0 = c_1^0 \text{and} a_1 \setminus c_1^1 \qquad\qquad = c_1^0 \setminus a_0 = c_1^1.$$

So we have cycles beginning $K(c_0^0, c_1^0, \ldots)$, and $K(a_1, c_1^1, \ldots)$. We can use the existing cycles to write:

$$\Delta = c_0^1 a_0 c_1^0 = c_0^1 c_1^1 a_0 = c_0^0 a_1 c_0^1$$

and by lemma 4.1.2 we can conclude that $c_0^1 \setminus c_1^1 = c_1^1$ and $c_0^0 \setminus a_1 = a_1$; so we have cycles $K(c_0^1, c_1^1, \ldots)$, and $K(c_0^0, a_1, \ldots)$.

To summarise we have a subset of atoms of G: $\{a_0, a_1, c_0^0, c_1^0, c_1^0, c_1^1\}$, satisfying cycles:

$$\begin{aligned}
C_{a_0} ={}_{a_1} C &= K(\ldots, c_0^0, c_1^0, \ldots) \\
C_{a_1} ={}_{a_0} C &= K(\ldots, c_0^1, c_1^1, \ldots) \\
{}_{c_1^0}C &= K(a_0, c_0^0, c_0^1) \\
C_{c_0^1} &= K(a_0, c_1^0, c_1^1) \\
{}_{c_0^1}C &= K(\ldots, c_0^0, a_1, \ldots) \\
C_{c_0^0} ={}_{c_{c_1^1}} C &= K(a_1, c_0^1, c_1^0) \\
C_{c_1^0} &= K(\ldots, a_1, c_1^1, \ldots)
\end{aligned}$$

That is, we have for $i = 1$ a set of atoms $B_i = \{a_m, c_l^m \mid m \in \{0,1\}, l = 0,1,\ldots i\}$ which satisfy cycles:

$$
\begin{aligned}
C_{a_0} =_{a_1} C &= K(\ldots, c_0^0, c_1^0, \ldots, c_i^0, \ldots) \\
C_{a_1} =_{a_0} C &= K(\ldots, c_0^1, c_1^1, \ldots, c_i^1, \ldots) \\
{}_{c_1^0}C &= K(a_0, c_0^0, c_0^1) \\
C_{c_{l-1}^1} =_{c_{l+1}^0} C &= K(a_0, c_l^0, c_l^1), \text{for } l = 1, 2, \ldots, i-2, i-1 \\
C_{c_{i-1}^1} &= K(a_0, c_i^0, c_i^1) \\
{}_{c_0^1}C &= K(\ldots, c_0^0, a_1, \ldots) \\
C_{c_l^0} =_{c_{c_{l+1}^1}} C &= K(a_1, c_l^1, c_{1+1}^0), \text{for } l = 0, 1, \ldots, i-2, i-1 \\
C_{c_i^0} &= K(\ldots, a_1, c_i^1, \ldots)
\end{aligned}
$$

Suppose we have, for some $j \in \mathbb{N}$, $j \geq 1$, that B_j is a set of distinct atoms satisfying the cycles listed.

We first consider the cube condition on triple (a_0, c_j^0, c_{j-1}^1), and find that $c_j^0 \setminus c_{j-1}^0 = c_j^1 \setminus a_1$. Call this atom c_{j+1}^0, we check whether it is possible that $c_{j+1}^0 \in B_j$:

$$
\begin{aligned}
&c_{j+1}^0 = a_0 \Rightarrow c_0^0 \setminus a_0 = c_1^0, \text{but } c_0^0 \setminus a_0 = c_0^1 \\
&c_{j+1}^0 = a_1 \Rightarrow c_0^0 \setminus a_1 = c_1^0, \text{but } c_0^0 \setminus a_1 = a_1 \\
&c_{j+1}^0 = c_l^0 \Rightarrow K(c_0^0, \ldots, c_{l-1}^0, c_l^0, \ldots, c_j^0, c_l^0, \ldots), \text{so } c_j^0 = c_{l-1}^0; \text{for } l = 1, 2, \ldots j \\
&c_{j+1}^0 = c_l^1 \Rightarrow a_1 \setminus c_l^1 = c_j^1, \text{but } a_1 \setminus c_l^1 = c_l^1 \text{for } l = 0, 1, \ldots j-1 \\
&c_{j+1}^0 = c_j^1 \Rightarrow K(a_1, c_j^1, c_j^1), \text{so } a_1 = c_j^1
\end{aligned}
$$

So either $c_j^0 \setminus c_{j-1}^0 \notin B_j$, or $c_j^0 \setminus c_{j-1}^0 = c_0^0$. We assume the former, and put $A_1 := B_j \cup \{c_{j+1}^0\}$, and note that this gives us cycles:

$$
\begin{aligned}
C_{a_0} =_{a_1} C &= K(\ldots, c_0^0, c_1^0, \ldots, c_j^0, c_{j+1}^0, \ldots) \\
C_{c_j^0} &= K(a_1, c_j^1, c_{j+1}^0) \\
C_{c_{j-1}^1} =_{c_{j+1}^0} C &= K(a_0, c_j^0, c_j^1)
\end{aligned}
$$

We proceed to show that under this assumption we find ourselves with a distinct set of atoms B_{j+1} with the associated cycles.

1. From the triple (a_1, c_j^1, c_j^0) we find that $c_j^1 \setminus c_{j-1}^1 = c_{j+1}^0 \setminus a_0$ and we call this atom c_{j+1}^1. It is a simple matter to show that $c_{j+1}^1 \notin A_1$, to avoid repetition we leave this to the reader.

So we put $A_2 = A_1 \cup \{c^1_{j+1}\}$, and note that we necessarily have cycles:

$$\begin{aligned} C_{a_1} =_{a_0} C &= K(\ldots, c^1_0, c^1_1, \ldots, c^1_j, c^1_{j+1}, \ldots) \\ C_{c^1_j} &= K(a_0, c^0_{j+1}, c^1_{j+1}) \\ C_{c^0_j} =_{c^1_{j+1}} C &= K(a_1, c^1_j, c^0_{j+1}) \end{aligned}$$

2. The cube condition on the triple (a_0, c^0_{j+1}, c^1_j) implies that $a_1 \setminus c^1_{j+1} = c^1_{j+1}$, so we have the cycle $C^0_{j+1} = K(\ldots, a_1, c^1_{j+1}, \ldots)$.

So by this induction we have, for $i = 0$, a set of distinct atoms $C_i = \{a_m, c^m_l \mid m \in \{0,1\},\ l = i, i+1, \ldots\}$; satisfying cycles

$$\begin{aligned} C_{a_0} =_{a_1} C &= K(\ldots, c^0_i, c^0_{i+1}, \ldots) \\ C_{a_1} =_{a_0} C &= K(\ldots, c^1_i, c^1_{i+1}, \ldots) \\ {}_{c^0_{i+1}}C &= K(a_0, c^0_i, c^1_i) \\ C_{c^1_{l-1}} =_{c^0_{l+1}} C &= K(a_0, c^0_l, c^1_l), \text{for } l = i+1, i+2, \ldots \\ {}_{c^1_i}C &= K(\ldots, c^0_i, a_1, \ldots) \\ C_{c^0_l} =_{c_{c^1_{l+1}}} C &= K(a_1, c^1_l, c^0_{1+1}), \text{for } l = i, i+1, \ldots \end{aligned}$$

Given distinct atoms C_j and associated cycles for some $j \leq 0$, we show that we necessarily have the same for $j-1$

1. The cube condition on the triple (a_1, c^1_j, c^0_j) implies $c^1_j \setminus (a_1 \setminus c^0_j) = c^1_{j+1}$. We call this atom $c^1_{j-1} = a_1 \setminus c^0_j$, and leave it to the reader to check that c^1_{j-1} is not any of the elements of C_j. So we put $C'_j = C_j \cup \{c^1_{j-1}\}$, and note that we have cycles:

$$\begin{aligned} C_{a_1} =_{a_0} C &= K(\ldots, c^1_{j-1}, c^1_j, c^1_{j+1} \ldots) \\ {}_{c^1_j}C &= K(a_1, c^1_{j-1}, c^0_j) \\ C^1_{j-1} =_{c^0_{j+1}} &= K(a_0, c^0_j, c^1_j) \end{aligned}$$

2. The factorisation $\Delta = c^1_0 a_0 c^0_1 = c^1_{j-1} a_0 c^0_j$ implies $c^1_{j-1} \setminus a_0 = a_0$, and thus the cycle ${}_{c^0_j}C = K(\ldots, c^1_{j-1}, a_0, \ldots)$.
3. The cube condition on (a_0, c^1_{j-1}, c^0_j) implies $c^0_j \setminus (a_0 \setminus c^1_{j-1}) = c^0_{j+1}$. We call this atom $c^0_{j-1} = a_0 \setminus c^1_{j-1}$, and one may check that $c^0_{j-1} \notin C'_j$. So we put $C''_j = C'_j \cup \{c^0_{j-1}\}$,

and note the implied cycles:

$$\begin{aligned} C_{a_0} =_{a_1} C &= K(\ldots, c^0_{j-1}, c^0_j, c^0_{j+1} \ldots) \\ {}_{c^0_j}C &= K(a_0, c^0_{j-1}, c^1_{j-1}) \\ C^0_{j-1} =_{c^1_j} &= K(a_1, c^1_{j-1}, c^0_j) \end{aligned}$$

4. The factorisation $\Delta = c^1_0 a_0 c^0_1 = c^0_{j-1} a_1 c^1_{j-1}$ gives $c^0_{j-1} \setminus a_1 = a_1$, and thus the cycles ${}_{c^1_{j-1}} = K(\ldots, c^0_{j-1}, a_1, \ldots)$.

So if we have none of the possible equivalences $c^0_j \setminus c^0_{j-1} = c^0_0$, then we have a group with cycle presentation:

$$\tilde{A}_2(\infty) = \left\langle \begin{array}{lr} a_m, & m \in \{0,1\} \\ c^m_l, & l \in \mathbb{Z}, m \in \{0,1\} \end{array} \;\middle|\; \begin{array}{lr} K(\ldots, c^m_{-1}, c^m_0, c^m_1, \ldots), & m \in \{0,1\} \\ K(a_m, c^m_l, c^{|m-1|}_{l+m}), & l \in \mathbb{Z}, m \in \{0,1\} \end{array} \right\rangle$$

Now by [20], the affine Artin group $\tilde{A}_2$ with generators c^1_0, a_0, c^0_1 and $\Delta = c^1_0 a_0 c^0_1$ meets all the assumptions of the statement, and it is clear that none of the possible relations $c^0_j \setminus c^0_{j-1} = c^0_0$ hold in $\tilde{A}_2$, so we may conclude that $\tilde{A}_2(\infty) = \tilde{A}_2$.

Alternatively if we have distinct atoms B_j, satisfying cycles as described above, and we take $c^0_j \setminus c^0_{j-1} = c^0_0$, then we have cycles:

$$\begin{aligned} C_{a_0} =_{a_1} C &= K(c^0_0, c^0_1, \ldots, c^0_j) \\ C_{a_1} =_{a_0} C &= K(c^1_0, c^1_1, \ldots, c^1_j) \\ C_{c^1_j} =_{c^0_1} C &= K(a_0, c^0_0, c^1_0) \\ C_{c^1_{l-1}} =_{c^0_{l+1}} C &= K(a_0, c^0_l, c^1_l), \text{for } l = 1, 2, \ldots, j-2, j-1 \\ C_{c^1_{j-1}} =_{c^0_0} C &= K(a_0, c^0_j, c^1_j) \\ C_{c^0_j} =_{c^1_0} C &= K(a_1, c^1_j, c^0_0) \\ C_{c^0_l} =_{c_{c^1_{l+1}}} C &= K(a_1, c^1_l, c^0_{1+1}), \text{for } l = 0, 1, \ldots, j-2, j-1 \end{aligned}$$

So that $G = \tilde{A}_2(j)$.

So $G = \tilde{A}_2(k)$ for some $k = 1, 2, \ldots$ or $G = \tilde{A}_2(\infty) = \tilde{A}_2$.

■

4.5 $\tilde{H}_2(k)$

We introduce a family of groups $\tilde{H}_2(k)$, $k \geq 1$, with cycle presentations.

Definition 4.5.1 *We denote by $\tilde{H}_2(k)$ the cycle presented group with set of generators:*

$$A_{\tilde{H}_2(k)} := \{a_m, b_l, c_l, d_l \mid l \in \mathbb{Z}_{2(k+2)}, m \in \{0,1\}\}$$

and relations given by cycles:

$$\begin{aligned}
&K(c_m, c_{m+2}, \ldots, c_{m+2(k+1)}), && m = 0,1\\
&K(a_{(l+1)-2\lfloor(l+1)/2\rfloor}, c_{l-1}, b_{l-1}, d_{l-1}, b_l, c_{l+2}), && l \in \mathbb{Z}_{2(k+2)}\\
&K(c_{l+2}, c_{l+1}, d_l), && l \in \mathbb{Z}_{2(k+2)}\\
&K(c_{l+2}, b_{l+1}), && l \in \mathbb{Z}_{2(k+2)}
\end{aligned}$$

We can easily write out the right and left adjacency sets of the atoms:

$$\begin{aligned}
R(a_m) = L(a_{|m-1|}) &= \{c_{m+2l} \mid l = 0, 1, \ldots k+2 \in \mathbb{Z}_{2(k+3)}\}, m \in \mathbb{Z}_{2(k+3)}\\
R(c_l) = L(c_{l+1}) &= \{a_{(l+1)-2\lfloor(l+1)/2\rfloor}, c_{l-1}, b_{l-1}, d_{l-1}, b_l, c_{l+2}\}, l \in \mathbb{Z}_{2(k+3)}\\
R(b_l) = L(b_{l+1}) &= \{c_{l+2}, c_{l+1}, d_l\}, l \in \mathbb{Z}_{2(k+3)}\\
R(d_l) = L(d_{l+1}) &= \{c_{l+2}, b_{l+1}\}, l \in \mathbb{Z}_{2(k+3)}
\end{aligned}$$

and conclude that the left and right adjacency sets of all atoms are the underlying sets of cycles. If for $x \in A_{\tilde{H}_2(k)}$ we call C_x the cycle of which $R(x)$ is the underlying set, then we have $xe(C_x) = ye(C_y)$ for any pair of atoms $x, y \in A_{\tilde{H}_2(k)}$.

So from Theorem 4.3.1, we can conclude that the group $\tilde{H}_2(k)$ is Bessis-Garside for each $k = 1, 2, \ldots$.

We have also the infinite case of this family $\tilde{H}_2(\infty)$ the cycle presented group with set of generators:

$$A = \{a_i, b_j, c_j, d_j \mid i \in \{0,1\}, j \in \mathbb{Z}\}$$

and cycles:

$$\begin{aligned}
C_{a_i} &=_{a_{|i-1|}} C = K(\ldots, c_{i-2}, c_i, c_{i+2}, \ldots)\\
C_{b_i} &=_{b_{i+1}} C = K(c_{i+2}, c_{i+1}, d_i)\\
C_{c_i} &=_{c_{i+1}} C = K(a_{(i+1)-2\lfloor(i+1)/2\rfloor}, c_{i-1}, b_{i-1}, d_{i-1}, b_i, c_{i+2})\\
C_{d_i} &=_{d_{i+1}} C = K(c_{i+1}, b_i)
\end{aligned}$$

which can similarly be seen to be Bessis-Garside.

Theorem 4.5.2 *Let G be a Bessis-Garside group, with standard triple (a, b, c), and with a, b, c, $b^{-1}ab$, $b^{-1}a^{-1}bab$, $b^{-1}a^{-1}b^{-1}abab$, $b^{-1}a^{-1}b^{-1}a^{-1}babab$, and $c^{-1}bc$ distinct atoms satisfying cycles $K(a, b, b^{-1}ab, b^{-1}a^{-1}bab, b^{-1}a^{-1}b^{-1}abab, b^{-1}a^{-1}b^{-1}a^{-1}babab)$, $K(b, c, c^{-1}bc)$,* *and $K(a,c)$. Then G is equal to $\tilde{H}_2(k)$ for some $k = 1, 2, \ldots$ or $\tilde{H}_2(\infty)$.*

Proof.

We begin by relabelling $a_0 := b^{-1}ab$, $b_0 := b^{-1}a^{-1}b^{-1}abab$, $b_1 := a$, $c_0 := b^{-1}a^{-1}bab$, $c_2 := c$, $c_3 := b$, $d_0 := b^{-1}a^{-1}b^{-1}a^{-1}babab$, and $d_1 := c^{-1}bc$; and put $A_0 := \{a_0, b_0, b_1, c_0, c_2, c_3, d_0, d_1\}$. We have then cycles:

$$
\begin{aligned}
{}_{c_2}C &= K(a_0, c_0, b_0, d_0, b_1, c_3)\\
C_{b_1} &= K(c_3, c_2, d_1)\\
{}_{d_1}C &= K(b_1, c_2)
\end{aligned}
$$

As one might expect this proof proceeds similarly to the proofs of theorems 4.2.7 and 4.4.2. As the method used has already been written out in some detail above, we list here only the triples over which the cube condition need be considered in order to construct the cycle structure of the group G. By considering in order the triples: (c_3, c_2, b_1), (a_0, c_2, c_3), (c_2, c_0, a_0), (b_0, c_2, c_0), (b_2, c_3, d_1), (c_3, c_1, c_2), (c_4, c_3, b_1), (b_3, c_4, c_3), and (c_5, c_4, b_2); we can construct a set of atoms $\{b_2 := d_1 \setminus b_1, c_4 := c_2 \setminus c_0, a_1 := c_2 \setminus b_0, c_1 := c_2 \setminus d_0, d_2 := b_2 \setminus c_2, c_5 := c_3 \setminus c_1, b_3 := d_2 \setminus a_0, d_3 := b_3 \setminus c_3, b_4 := c_4 \setminus a_0\}$ distinct from one another and from the atoms already mentioned in the standard triples.

Indeed we have for $i = 2$ a subset of distinct atoms:

$$B_i = \{a_0, a_1, c_0, c_1, \ldots, c_{2i+1}, b_0, b_1, \ldots, b_{2i}, d_0, d_1, \ldots, d_{2i-1}\}$$

which satisfy the cycles:

$$
\begin{aligned}
C_{a_0} =_{a_1} C &= K(\ldots, c_0, c_2, \ldots, c_{2i}, \ldots)\\
C_{a_1} =_{a_0} C &= K(\ldots, c_1, c_3, \ldots, c_{2i+1}, \ldots)\\
C_{b_l} =_{b_{l+1}} C &= K(c_{l+2}, c_{l+1}, d_l), \text{for } l = 0, \ldots, 2i-1\\
C_{c_0} =_{c_1} C &= K(\ldots, b_0, c_2, a_1, \ldots)\\
C_{c_l} =_{c_{l+1}} C &= K(a_{(l+1)-2\lfloor (l+1)/2 \rfloor}, c_{l-1}, b_{l-1}, d_{l-1}, b_l, c_{l+2}), \text{for } l = 1, \ldots 2i-1\\
C_{c_{2i}} =_{c_{2i+1}} C &= K(\ldots, a_0, c_{2i-1}, b_{2i-1}, d_{2i-1}, b_{2i}, \ldots)\\
C_{c_{2i+1}} &= K(\ldots, a_0, c_{2i}, b_{2i}, \ldots)\\
{}_{d_0}C &= K(c_1, b_0)\\
C_{d_l} =_{d_{l+1}} C &= K(c_{l+2}, b_{l+1}), \text{for } l = 0, \ldots, 2i-2\\
C_{d_{2i-1}} &= K(c_{2i+1}, b_{2i})
\end{aligned}
$$

We proceed by induction on i. Assume we have distinct atoms B_j and associated cycles for some $j \geq 2$.

From the cube condition on the triple $(c_{2j,a_0,c_{2j-2}})$, we have $(c_{2j} \setminus c_{2j-2}) \setminus b_{2j} = a_1$. One may show that if $c_{2j} \setminus c_{2j-2} \in B_j$ then $c_{2j} \setminus c_{2j-2} = c_0$.

We assume that $c_{2j} \setminus c_{2j-2} \neq c0$, and we put $c_{2j+2} := c_{2j} \setminus c_{2j-2}$. By considering the cube condition on the triples $(b_{2j}, c_{2j+1}, c_{2j})$, $(c_{2j+1}, c_{2j+2}, b_{2j-1})$, $(a_1, c_{2j+1}, c_{2j-1})$, $(b_{2j+1}, c_{2j+2}, c_{2j+1})$, and $(c_{2j+2}, c_{2j+3}, b_{2j})$; we can construct atoms $\{d_{2j} := b_{2j} \setminus c_{2j}, b_{2j+1} := c_{2j+1} \setminus a_1, c_{2j+3} := c_{2j+1} \setminus c_{2j-1}, d_{2j+1} := b_{2j+1} \setminus c_{2j+1}, b_{2j+2} := c_{2j+2} \setminus a_0\}$ distinct from one another, from the atoms of B_j, and from c_{2j+2}. We have then a set of distinct atoms B_{j+1}, which satisfy the associated cycles described above.

If we assume for all $m \geq 0$ that $c_{2m} \setminus c_{2m-2} \neq c_0$, then we have, for $i = 0$, a subset of distinct atoms $D_i = \{a_0, a_1, c_l, b_l, d_l \mid l = i, i+1, \ldots\}$ satisfying cycles:

$$\begin{aligned}
&C_{a_i} =_{a_{|i-1|}} C && = K(\ldots, c_i, c_{i+2}, \ldots)\\
&C_{a_{|i-1|}} =_{a_i} C && = K(\ldots, c_{i+1}, c_{i+3}, \ldots)\\
&C_{b_l} =_{b_{l+1}} C && = K(c_{l+2}, c_{l+1}, d_l), \text{for } l = i, i+1, \ldots\\
&C_{c_i} =_{c_{i+1}} C && = K(\ldots, b_i, c_{i+2}, a_{(i+1)-2\lfloor (i+1)/2 \rfloor}, \ldots)\\
&C_{c_l} =_{c_{l+1}} C && = K(a_{(l+1)-2\lfloor (l+1)/2 \rfloor}, c_{l-1}, b_{l-1}, d_{l-1}, b_l, c_{l+2}), \text{for } l = i+1, , i+2, \ldots\\
&{}_{d_i}C && = K(c_{i+1}, b_i)\\
&C_{d_l} =_{d_{l+1}} C && = K(c_{l+2}, b_{l+1}), \text{for } l = i, i+1, \ldots
\end{aligned}$$

We show that in this case $G = \tilde{H}_2(\infty)$, proceeding by induction on i. Suppose we have atoms D_k and associated cycles for some $k \leq 0$. By considering cube condition on the triples of atoms $(a_{k\bar{+}1}, c_{k+4}, c_{k+2})$, $(a_{k\bar{+}1}, c_{k-1}, c_{k+1})$, and $(b_{k-1}, c_{k-1}, c_{k+1})$; we can construct atoms $\{c_{k-1} := a_{k\bar{+}1} \setminus c_{k+2}, b_{k-1} := c_{k-1} \setminus a_{k\bar{+}1}, d_{k-1} := b_{k-1} \setminus c_{k-1}\}$ distinct from one another and from the atoms in D_k. We have then a set of distinct atoms D_{k-1}, which satisfy the associated cycles described above.

So if there is no $m \geq 1$ such that $c_{2m} \setminus c_{2m-2} = c_0$, then G has a cycle presentation with atoms $A = \{a_0, a_1, b_j, c_j, d_j \mid i \in \{0,1\}, j \in \mathbb{Z}\}$, and cycles:

$$\begin{aligned}
C_{a_i} =_{a_{|i-1|}} C &= K(\ldots, c_{i-2}, c_i, c_{i+2}, \ldots)\\
C_{b_i} =_{b_{i+1}} C &= K(c_{i+2}, c_{i+1}, d_i)\\
C_{c_i} =_{c_{i+1}} C &= K(a_{(i+1)-2\lfloor (i+1)/2 \rfloor}, c_{i-1}, b_{i-1}, d_{i-1}, b_i, c_{i+2})\\
C_{d_i} =_{d_{i+1}} C &= K(c_{i+1}, b_i)
\end{aligned}$$

for $i \in \mathbb{Z}$, that is exactly $G = \tilde{H}_2(\infty)$.

Returning to the subset of atoms B_j and the triple (c_{2j}, a_0, c_{2j-2}) we now suppose that $c_{2j} \setminus c_{2j-2} = c_0$, and we show that in this case we have $G = \tilde{H}_2(j-1)$. We need only consider the cube condition on the triples (c_{2j}, a_0, c_{2j-2}), $(b_{2j}, c_{2j+1}, c_{2j})$, $(c_{2j+1}, c_{2j}, b_{2j-1})$, and $(b_{2j+1}, c_0, c_{2j+1})$; to construct atoms $\{d_{2j} := b_{2j} \setminus c_{2j}, b_{2j+1} := c_{2j+1} \setminus a_1, d_{2j+1} := c_0 \setminus c_1\}$

We have then a set of atoms $\{a_m, b_l, c_l, d_l \mid l \in \mathbb{Z}_{2(j+1)}, m \in \{0,1\}\}$, which must satisfy cycles:

$$\begin{array}{ll}
K(c_m, c_{m+2}, \ldots, c_{m+2(k+1)}), & m = 0,1 \\
K(a_{(l+1)-2\lfloor(l+1)/2\rfloor}, c_{l-1}, b_{l-1}, d_{l-1}, b_l, c_{l+2}), & l \in \mathbb{Z}_{2(j+1)} \\
K(c_{l+2}, c_{l+1}, d_l), & l \in \mathbb{Z}_{2(j+1)} \\
K(c_{l+2}, b_{l+1}), & l \in \mathbb{Z}_{2(j+1)}
\end{array}$$

Thus if $c_{2j} \setminus c_{2j-2} = c_0$ we have exactly $G = \tilde{H}_2(j-1)$ as stated.

■

As a corollary to the preceding theorem, we can see that the group $\tilde{H}_2$ is not itself Bessis-Garside.

Corollary 4.5.3 *The group $\tilde{H}_2$ is Bessis of rank 3, but not Bessis-Garside of rank 3.*

Proof. From the definition of $\tilde{H}_2$ given at the beginning of this chapter we can see that it admits cycles $K(a, b, b^{-1}ab, b^{-1}a^{-1}bab, b^{-1}a^{-1}b^{-1}abab, b^{-1}a^{-1}b^{-1}a^{-1}babab)$, $K(b, c, c^{-1}bc)$, and $K(a, c)$. Therefore by the preceding theorem if it were Bessis-Garside it would have to be one of $\tilde{H}_2(k)$ where $k = 1, 2, \ldots$, or $\tilde{H}_2(\infty)$ whereas the relations of the cycle $(a_0, c_2, b_2, d_2, b_3, c_5)$ (that is $(b^{-1}ab, c, c^{-1}b^{-1)abc,\ldots)}$ under the identification $a_0 = b^{-1}ab$, $c_2 = c$ etc.) does not hold in $\tilde{H}_2$. ■

4.6 Cycle presentations of spherical Artin groups of rank 3

We know from [4] that all spherical Artin groups are Bessis-Garside. Accordingly we can conclude from Theorem 4.3.1 that all spherical Artin groups of rank 3 have a cycle presentation meeting the conditions of that theorem. We provide such a presentation for each of the three rank 3 spherical Artin groups, and note that the existence of such presentations comprises an alternative proof that these groups admit Bessis-Garside structure.

Example 4.6.1 *The Artin group commonly denoted A_3, is simply the braid group on 2 strands, which has presentation:*

$$A_3 = \left\langle a, b, c \mid \begin{array}{l} aba = bab; \\ bcb = cbc; \\ ac = ca \end{array} \right\rangle$$

A_3 is isomorphic to the cycle presented group:

$$G = \left\langle \begin{array}{l} \alpha_i, i \in \mathbb{Z}_4 \\ \beta_j, j \in \mathbb{Z}_2 \end{array} \mid \begin{array}{l} K(\beta_{\bar{i}}, \alpha_i, \alpha_{i-1}), i \in \mathbb{Z}_4 \\ K(\alpha_i, \alpha_{i+2}), i = 0, 1 \in \mathbb{Z}_4 \end{array} \right\rangle$$

With isomorphism given by $\phi : A_3 \to G$ given by $a \mapsto \alpha_0$, $b \mapsto \alpha_3$, $c \mapsto \alpha_2$. To see that this is an isomorphism note that the cycles $K(\alpha_0, \alpha_3, \beta_0)$, $K(\alpha_3, \alpha_2, \beta_1)$, and $K(\alpha_0, \alpha_2)$ are equivalent to the relations $\alpha_0\alpha_3\alpha_0 = \alpha_3\alpha_0\alpha_3$, $\alpha_3\alpha_2\alpha_3 = \alpha_2\alpha_3\alpha_2$, and $K(\alpha_0\alpha_2 = \alpha_2\alpha_0)$ respectively, so that ϕ is indeed a homomorphism. We can write $\beta_0 = \alpha_3^{-1}\alpha_0\alpha_3$, $\beta_1 = \alpha_2^{-1}\alpha_3\alpha_2$, and $\alpha_1 = \alpha_2^{-1}\alpha_3^{-1}\alpha_0\alpha_3\alpha_2$, to see that ϕ is surjective. Finally we demonstrate that the remaining cycle relations are a consequence of the Artin relations, so that ϕ is injective:

$$
\begin{aligned}
K(\alpha_1, \alpha_3) \Leftrightarrow & \alpha_3(\alpha_2^{-1}\alpha_3^{-1}\alpha_0\alpha_3\alpha_2) \\
&= \alpha_2^{-1}\alpha_3^{-1}\alpha_2\alpha_0\alpha_3\alpha_2 \\
&= \alpha_2^{-1}\alpha_3^{-1}\alpha_0\alpha_2\alpha_3\alpha_2 \\
&= (\alpha_2^{-1}\alpha_3^{-1}\alpha_0\alpha_3\alpha_2)\alpha_3 \\
K(\alpha_1, \alpha_0, \beta_1) \Leftrightarrow &= (\alpha_2^{-1}\alpha_3^{-1}\alpha_0\alpha_3\alpha_2)\alpha_0 \\
&= \alpha_2^{-1}\alpha_3^{-1}\alpha_0\alpha_3\alpha_0\alpha_2 \\
&= \alpha_2^{-1}\alpha_0\alpha_3\alpha_2 \\
&= \alpha_0(\alpha_2^{-1}\alpha_3\alpha_2) \\
&= (\alpha_2^{-1}\alpha_3\alpha_2)(\alpha_2^{-1}\alpha_3^{-1}\alpha_0\alpha_3\alpha_2) \\
K(\alpha_2, \alpha_1, \beta_0) \Leftrightarrow &= \alpha_2(\alpha_2^{-1}\alpha_3^{-1}\alpha_0\alpha_3\alpha_2) \\
&= (\alpha_3^{-1}\alpha_0\alpha_3)\alpha_2 \\
&= \alpha_3^{-1}\alpha_2^{-1}\alpha_0\alpha_2\alpha_3\alpha_2 \\
&= \alpha_3^{-1}\alpha_2^{-1}\alpha_3^{-1}\alpha_3\alpha_0\alpha_3\alpha_2\alpha_3 \\
&= \alpha_2^{-1}\alpha_3^{-1}\alpha_2^{-1}\alpha_0\alpha_3\alpha_0\alpha_2\alpha_3 \\
&= \alpha_2^{-1}\alpha_3^{-1}\alpha_0\alpha_2^{-1}\alpha_3\alpha_2\alpha_0\alpha_3 \\
&= (\alpha_2^{-1}\alpha_3^{-1}\alpha_0\alpha_3\alpha_2)(\alpha_3^{-1}\alpha_0\alpha_3).
\end{aligned}
$$

So $G \cong A_3$.
We may note that:

$$
\begin{aligned}
R(\alpha_{i-1}) = L(\alpha_i) &= \{\beta_{\bar{i}}, \alpha_{i+2}, \alpha_{i+1}\} \text{for } i \in \mathbb{Z}_4 \\
R(\beta_{\bar{i}}) = L(\beta_{\bar{i+1}}) &= \{\alpha_i, \alpha_{i+2}\} \text{for } i \in \mathbb{Z}_4
\end{aligned}
$$

which by Theorem 4.3.1 confirms that A_3 is Bessis-Garside.
We can also note that this group is the same as $\tilde{A}_2(1)$ from earlier in this chapter.

Example 4.6.2 *The Artin group commonly denoted B_3, has presentation:*

$$
B_3 = \left\langle a, b, c \;\middle|\; \begin{array}{l} abab = baba; \\ bcb = cbc; \\ ac = ca \end{array} \right\rangle
$$

B_3 is isomorphic to the cycle presented group:

$$G = \left\langle \begin{array}{l} \alpha_i \\ \beta_i, i \in \mathbb{Z}_3 \\ \gamma_i \end{array} \;\middle|\; \begin{array}{l} K(\alpha_i, \beta_{i+2}, \gamma_i, \beta_i) \\ K(\alpha_i, \alpha_{i+2}, \gamma_{i+1}), i \in \mathbb{Z}_3 \\ K(\alpha_i, \beta_{i+1}) \end{array} \right\rangle$$

With isomorphism given by $\phi : B_3 \to G$ given by $a \mapsto \beta_0$, $b \mapsto \alpha_0$, $c \mapsto \alpha_2$.
We may note that:

$$\begin{aligned} R(\alpha_{i+1}) = L(\alpha_{i+2}) &= \{\alpha_i, \beta_{i+2}, \gamma_i, \beta_i\} \text{for } i \in \mathbb{Z}_3 \\ R(\beta_i) = L(\beta_{i+1}) &= \{\alpha_i, \alpha_{i+2}, \gamma_{i+1}\} \text{for } i \in \mathbb{Z}_3 \\ R(\gamma_{i+1}) = L(\gamma_{i+2}) &= \{\alpha_i, \beta_{i+1}\} \text{for } i \in \mathbb{Z}_3 \end{aligned}$$

which by Theorem 4.3.1 confirms that B_3 is Bessis-Garside.

Example 4.6.3 *The Artin group commonly denoted H_3, has presentation:*

$$H_3 = \left\langle a, b, c \;\middle|\; \begin{array}{l} ababa = babab; \\ bcb = cbc; \\ ac = ca \end{array} \right\rangle$$

H_3 is isomorphic to the cycle presented group:

$$G = \left\langle \begin{array}{l} \alpha_i \\ \beta_i, i \in \mathbb{Z}_5 \\ \gamma_i \end{array} \;\middle|\; \begin{array}{l} K(\alpha_i, \alpha_{i+2}, \beta_{i-1}, \gamma_i, \beta_i) \\ K(\alpha_i, \alpha_{i+4}, \gamma_{i+1}), i \in \mathbb{Z}_5 \\ K(\alpha_i, \beta_{i+1}) \end{array} \right\rangle$$

With isomorphism given by $\phi : H_3 \to G$ given by $a \mapsto \beta_0$, $b \mapsto \alpha_0$, $c \mapsto \alpha_4$.
We may note that:

$$\begin{aligned} R(\alpha_{i-2}) = L(\alpha_{i-1}) &= \{\alpha_{i+2}, \beta_{i-1}, \gamma_i, \beta_i, \alpha_i\} \text{for } i \in \mathbb{Z}_5 \\ R(\beta_i) = L(\beta_{i+1}) &= \{\alpha_i, \alpha_{i+4}, \gamma_{i+1}\} \text{for } i \in \mathbb{Z}_5 \\ R(\gamma_{i+1}) = L(\gamma_{i+2}) &= \{\alpha_i, \beta_{i+1}\} \text{for } i \in \mathbb{Z}_5 \end{aligned}$$

which by Theorem 4.3.1 confirms that H_3 is Bessis-Garside.

4.7 Representations over finite fields

Lemma 4.7.1 *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ d & 0 & 1 \end{pmatrix} B = \begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & e & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}$$

and define

$$p_k = \begin{cases} ab & \text{if } k \in 2\mathbb{Z}; \\ 1 & \text{otherwise.} \end{cases}$$

$$q_k = \begin{cases} cd & \text{if } k \in 2\mathbb{Z}; \\ 1 & \text{otherwise.} \end{cases}$$

$$r_k = \begin{cases} ef & \text{if } k \in 2\mathbb{Z}; \\ 1 & \text{otherwise.} \end{cases}$$

and let w_k *be the sequence given by* $w_0 = 0$, $w_1 = 1$, *and* $w_{k+1} = w_{k-1} + p_k w_k$; y_k *be that given by* $y_0 = 0$, $y_1 = 1$, *and* $y_{k+1} = y_{k-1} + q_k y_k$; *and* z_k *be the sequence given by* $z_0 = 0$, $z_1 = 1$, *and* $z_{k+1} = z_{k-1} + r_k z_k$
Then

$$(A,B;k)(A,B;k-1)^{-1} = \begin{pmatrix} 1+(-1)^{k+1}abw_{k-1}w_k & b+(-1)^k p_k b w_{k-2}w_k & 0 \\ (-1)^{k+1}p_{k-1}aw_k^2 & 1+(-1)^k abw_{k-1}w_k & 0 \\ (-1)^{k+1}w_k(p_k dw_k + aew_{k-1}) & e+(-1)^k w_k(p_k ew_{k-2}+bdw_{k-1}) & 1 \end{pmatrix}$$

$$(B,C;k)(B,C;k-1)^{-1} = \begin{pmatrix} 1 & (-1)^{k+1}y_k(q_k by_k + ecy_{k-1}) & c+(-1)^k y_k(q_k cy_{k-2}+bfy_{k-1}) \\ 0 & 1+(-1)^{k+1}efy_{k-1}y_k & f+(-1)^k q_k fy_{k-2}y_k \\ 0 & (-1)^{k+1}q_k ey_k^2 & 1+(-1)^k efy_{k-1}y_k \end{pmatrix}$$

$$(A,C;k)(A,C;k-1)^{-1} = \begin{pmatrix} 1+(-1)^{k+1}cdz_{k-1}z_k & 0 & c+(-1)^k r_k cz_{k-2}z_k \\ (-1)^{k+1}z_k(r_k az_k + dfz_{k-1}) & 1 & f+(-1)^k z_k(r_k fz_{k-2}+acz_{k-1}) \\ (-1)^{k+1}r_k dz_k^2 & 0 & 1+(-1)^k cdz_{k-1}z_k \end{pmatrix}$$

Proof. First note that $(A,B;k)(A,B;k-1)^{-1} = AB((A,B;k-1)(A,B;k-2)^{-1})^{-1}$ is:

$$\begin{pmatrix} 1 & b & 0 \\ a & 1+ab & 0 \\ d & bd+e & 1 \end{pmatrix} \begin{pmatrix} 1+(-1)^{k+1}abw_{k-2}w_{k-1} & -b+(-1)^k p_{k-1}w_{k-3}w_{k-1} & 0 \\ (-1)^{k+1}p_{k-1}aw_{k-1}^2 & 1+(-1)^k abw_{k-2}w_{k-1} & 0 \\ (-1)^{k+1}w_{k-1}(p_{k-1}dw_{k-1}+aew_{k-2}) & e+(-1)^k w_{k-1}(p_{k-1}w_{k-3}e+w_{k-2}bd) & 1 \end{pmatrix}$$

From Theorem 3.2.4, we need only show $[(A,B;k)(A,B;k-1)^{-1}]_{3,1} = (-1)^{k+1}w_k(p_k dw_k + aew_{k-1})$ and $[(A,B;k)(A,B;k-1)^{-1}]_{3,2} = e+(-1)^k w_k(p_k w_{k-2}e + w_{k-1}bd)$.
If k even:

$$\begin{aligned}[(A,B;k)(A,B;k-1)^{-1}]_{3,1} &= d - abdw_k w_{k-1} - aew_{k-1}^2 - dw_{k-1}^2 - aew_{k-2}w_{k-1} \\ &= d(1-abw_k w_{k-1} - w_{k-1}^2) - ae(w_{k-1}(w_{k-2}+w_{k-1})) \\ &= -abdw_k^2 - aew_{k-1}w_k \end{aligned}$$

and

$$
\begin{aligned}
[(A,B;k+1)(A,B;k)^{-1}]_{3,1} &= d + abdw_{k+1}w_k + a^2bew_k^2 + abdw_k^2 - aew_{k-1}w_k \\
&= d(1 + abw_{k+1}w_k + abw_k^2) + ae(w_k(w_{k-1} + abw_k)) \\
&= dw_{k+1}^2 + aew_{k+1}w_k
\end{aligned}
$$

Similarly if k is even:

$$
\begin{aligned}
[(A,B;k)(A,B;k-1)^{-1}]_{3,2} &= e + (bdw_{k-1}^2 + abew_{k-2}w_{k-1}) - e + w_{k-1}(ew_{k-3} + bdw_{k-2}) \\
&= bdw_{k-1}(w_{k-2} + w_{k-1}) + ew_{k-1}(w_{k-3} + abw_{k-2}) \\
&= bdw_{k-1}w_k + ew_{k-1}^2 \\
&= bdw_{k-1}w_k + e(1 + abw_k^2 - abw_{k-1}w_k) \\
&= e + w_k(bdw_{k-1} + abew_{k-2})
\end{aligned}
$$

and

$$
\begin{aligned}
[(A,B;k+1)(A,B;k)^{-1}]_{3,2} &= e - (ab^2dw_k^2 + abew_{k-1}w_k) - e - w_k(abew_{k-2} + bdw_{k-1}) \\
&= -bdw_k(w_{k-1} + abw_k) - abew_k(w_{k-2}w_{k-1}) \\
&= -bdw_kw_{k+1} - abew_k^2 \\
&= -bdw_kw_{k+1} - e(w_{k+1}^2 - abw_kw_{k+1} - 1) \\
&= e - w_{k+1}(bdw_k + ew_{k-1})
\end{aligned}
$$

so for all k, $[(A,B;k)(A,B;k-1)^{-1}]_{3,1} = 0(\text{mod } w_k)$ and $[(A,B;k+1)(A,B;k)^{-1}]_{3,1} = d(\text{mod } w_k)$, $[(A,B;k)(A,B;k-1)^{-1}]_{3,2} = e(\text{mod } w_k)$ and $[(A,B;k+1)(A,B;k)^{-1}]_{3,2} = 0(\text{mod } w_k)$.
The descriptions of $(B,C;k)(B,C;k-1)^{-1}$ and $(A,C;k)(A,C;k-1)^{-1}$ follow by symmetry, and therefore the result holds. ∎

It is obvious that we can use this lemma to construct representations of Artin groups of rank 3 over finite fields.

Example 4.7.2 *Let*

$$
A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}
$$

then the map ρ from $\tilde{C}_2$ to $Gl(3,\mathbb{F}_3)$ given by sending $a \mapsto A$, $b \mapsto B$ and $c \mapsto C$ is a representation.

Chapter 5

Bessis-Garside groups and cycle length

In this chapter we consider Bessis-Garside groups of rank 3, in particular with presentations of the form given in theorem 4.3.1. In the first section we give a classification of all such presentations where all cycles have length at most 4.

In the rest of the chapter we look at the special case of such presentations where all cycles have the same length.

We place restrictions on the length of the cycles arising in the presentations of finitely-generated Bessis-Garside groups.

5.1 Bessis-Garside groups of cycle length at most k

The case where $k = 2$ is trivial, we have only the group $\mathbb{Z}^3$.

Lemma 5.1.1 *Let G be a Bessis-Garside group of rank 3, with atoms A such that $l(a) = 2$ for all $a \in A$, then $G = I_2(2) \times \mathbb{Z} = \mathbb{Z}^3$*

Proof. First note that $\Delta = x_1x_2x_3$, so by lemma 4.1.2 we have:

$$x_1 \setminus x_2 = x_2$$
$$x_2 \setminus x_3 = x_3$$
$$x_1 \setminus x_3 = x_3$$

Since all cycles have length 2, this gives us:

$$\begin{aligned}
C_{x_1} &=_{x_1} C &&= K(x_2, x_3)\\
C_{x_2} &=_{x_2} C &&= K(x_1, x_3)\\
C_{x_3} &=_{x_3} C &&= K(x_1, x_2)
\end{aligned}$$

So $A = p_1((x_1, x_2, x_3).B_3) = \{x_1, x_2, x_3\}$, and:

$$G = \langle x_i, i = 1, 2, 3 \mid x_i x_j = x_j x_i, \text{for } i \neq \rangle$$

that is, $G = I_2(2) \times \mathbb{Z} = \mathbb{Z}^3$ ■

We note the following useful lemma.

Lemma 5.1.2 *Let G_H be the Bessis group associated to the subgroup H of B_3, then if $H^h := \{hgh^{-1} \mid g \in H\}$ is the subgroup of B_3 given by conjugating H by some $h \in B_3$, then the Bessis group associated to H^h, G_{H^h} is isomorphic to G_H.*

Proof. Put $N := \langle x(x.g)^{-1}, \text{for } g \in H\rangle$, and $N^h := \langle x(x.g)^{-1}, \text{for } g \in H^h\rangle$, so that $G = F_3/N$, $G_{H^h} = F_3/N^h$. Let $\Theta : G_H \to G_{H^h}$ be given by $x_i N \mapsto x_i.h^{-1}N^h$. If $xN = yN$ in G, then there exists $g \in H$ such that $x.g = y$, which occurs if and only if $x.h^{-1}(hgh^{-1}) = y.h^{-1}$, that is if and only if $x.h^{-1}N^h = y.h^{-1}N^h$. So Θ is an injective homomorphism. Since $x_i.hN \in G$, we have $x_i.hh^{-1}N^h = x_i N^h \in \Theta(G)$, so Θ is surjective. ■

We use this to drastically reduce the number of cases to be considered in the classification to follow.

Definition 5.1.3 *Let G be a Bessis group of rank 3, with (regular) standard triple (a, b, c). By lemma 4.1.2, we have cycles $C_1 = K(\ldots, a, b, \ldots)$, $C_2 = K(\ldots, b, c, \ldots)$, $C_3 = K(\ldots, a, c, \ldots)$. We define:*

$$\tau_G := (l(C_1), l(C_2), l(C_3))$$

Lemma 5.1.4 *If G is a Bessis-Garside group of rank 3, with $\tau_G = (j, k, l)$, then there exists an isomorphic Bessis-Garside group, G', with $\tau_{G'} = (k, l, j)$*

Proof. We apply lemma 5.1.2 with $h = \sigma_1^{-1}\sigma_2^{-1}$, putting $G = G_H$ for some H, and $G' = G_{H^{\sigma_1^{-1}\sigma_2^{-1}}}$. Since $\tau_G = (j, k, l)$, we have:

$$\begin{aligned}
&l(K(\ldots, c, c^{-1}ac, \ldots)) = \\
&l(K(\ldots, a, c, \ldots)) = j, \\
&l(K(\ldots, c, c^{-1}bc, \ldots)) \\
&l(K(\ldots, b, c, \ldots)) = l, \text{and} \\
&l(K(\ldots, c^{-1}ac, c^{-1}bc, \ldots)) \\
&l(K(\ldots, a, b, \ldots)) = k
\end{aligned}$$

So $G' \cong G$, and $\tau_{G'} = (k, l, j)$ as required. ■

When $k = 3$ we have, in addition to the group with all cycles length 2, three more groups: $I_2(3) \times \mathbb{Z}$, $\tilde{A}_2(1)$ (which is the braid group on 3 strands), and $\tilde{A}_2(2)$.

Lemma 5.1.5 *Let (G, Δ) be a Bessis-Garside pair of rank 3, and consider the cycle presentation of G satisfying Condition 4.3.2 which exists as a result of Theorem 4.3.1. Suppose that $l(a) \leq 3$ for all $a \in A$, then G is isomorphic to one of $I_2(k) \times \mathbb{Z}$ with $k = 2, 3$ or $\tilde{A}_2(k)$ with $k = 1, 2$.*

Proof. By lemma 5.1.4 we need only consider 4 cases:

1. $\tau = (2, 2, 2)$.
2. $\tau = (3, 3, 3)$.
3. $\tau = (3, 2, 2)$.
4. $\tau = (3, 3, 2)$.

Up to permutation on the order of the standard triple. In the first case we have $G \cong I_2(2) \times (Z)$, by lemma 5.1.1; and in the second $G \cong \tilde{A}_2(k)$ for $k = 1, 2$ by theorem 4.4.2. Let us then consider case 3. From the cube condition on the triple (x_1, x_2, x_3) we see that $(x_2^{-1}x_1x_2)x_3 = x_3(x_2^{-1}x_1x_2)$, so that we have cycle $K(x_2^{-1}x_1x_2, x_3)$. This along with the cycles $K(x_1, x_2, x_2^{-1}x_1x_2)$, $K(x_2, x_3)$, and $K(x_1, x_3)$ makes it clear that $A = \{x_1, x_2, x_3, x_2^{-1}x_1x_2\}$, and that $G \cong I_2(3) \times \mathbb{Z}$. Finally in case 4 we begin with known distinct atoms $\{x_1, x_2, x_3, x_2^{-1}x_1x_2, x_3^{-1}x_2x_3\}$, and cycles $K(x_1, x_2, x_2^{-1}x_1x_2)$, $K(x_2, x_3, x_3^{-1}x_2x_3)$, and $K(x_1, x_3)$. From the cube condition on the triple (x_1, x_2, x_3) we have $x_3 \setminus (x_2^{-1}x_1x_2) = (x_3^{-1}x_2x_3) \setminus x_1$, and we confirm that this atom is not one of those already known:

$$\begin{aligned}
x_3 \setminus (x_2^{-1}x_1x_2) = x_1 &\Rightarrow K(\ldots, x_1, x_3^{-1}x_2x_3, x_1, \ldots), \text{so } x_1 = x_3^{-1}x_2x_3 \\
x_3 \setminus (x_2^{-1}x_1x_2) = x_2 &\Rightarrow x_1 \setminus x_2 = x_3^{-1}x_2x_3, \text{but } x_1 \setminus x_2 = x_2 \\
x_3 \setminus (x_2^{-1}x_1x_2) = x_3 &\Rightarrow x_1 \setminus x_3 = x_3^{-1}x_2x_3, \text{but } x_1 \setminus x_3 = x_3 \\
x_3 \setminus (x_2^{-1}x_1x_2) = x_2^{-1}x_1x_2 &\Rightarrow x_1 \setminus x_2^{-1}x_1x_2 = x_3^{-1}x_2x_3, \text{but } x_1 \setminus x_2^{-1}x_1x_2 = x_2 \\
x_3 \setminus (x_2^{-1}x_1x_2) = x_3^{-1}x_2x_3 &\Rightarrow K(\ldots, x_1, x_3^{-1}x_2x_3, x_3^{-1}x_2x_3, \ldots), \text{so } x_1 = x_3^{-1}x_2x_3
\end{aligned}$$

So we have $x_3 \setminus (x_2^{-1}x_1x_2)$ an atom distinct from those listed above, and cycles $K(\ldots, x_1, x_3^{-1}x_2x_3, x_3 \setminus (x_2^{-1}x_1x_2), \ldots)$, and $K(\ldots, x_2^{-1}x_1x_2, x_3, x_3 \setminus (x_2^{-1}x_1x_2), \ldots)$. We have ${}_{x_3}C = K(x_1, x_2, x_2^{-1}x_1x_2)$, so that $l(x_3) = 3$, and since $x_1, x_3^{-1}x_2x_3 \in R(x_3)$, we may conclude: $C_{x_3} = K(x_1, x_3^{-1}x_2x_3, x_3 \setminus (x_2^{-1}x_1x_2))$. Similarly ${}_{x_2}C = C_{x_3}$ gives $l(x_2) = 3$, and since $x_2^{-1}x_1x_2, x_3 \in R(x_2)$ we may conclude: $C_{x_2} = K(x_2^{-1}x_1x_2, x_3, x_3 \setminus (x_2^{-1}x_1x_2))$. Consider the cube condition on triple $(x_1, x_3, x_3^{-1}x_2x_3)$ we see that $x_2(x_3 \setminus (x_2^{-1}x_1x_2)) = (x_3 \setminus (x_2^{-1}x_1x_2))x_2$, so that we have cycle $K(x_2, x_3 \setminus (x_2^{-1}x_1x_2))$. This suffices to show that $A = \{x_1, x_2, x_3, x_2^{-1}x_1x_2, x_3^{-1}x_2x_3, x_3 \setminus (x_2^{-1}x_1x_2)\}$, and $G \cong \tilde{A}_2(1)$. This completes the proof. ■

We give cycle presentations for some Bessis-Garside groups of rank 3 which have all cycles of length at most 4. These will comprise all the exceptional cases in the classification of such groups.

Definition 5.1.6

P_{14}*is the group with cycle presentation given by atoms* :

$A_{P_{14}} := \{\alpha_i, \beta_i \ i \in \mathbb{Z}_7\}$

and cycles :

$C_{\alpha_i} =_{\alpha_{i+1}} C = K(\alpha_{i+3}, \beta_{i+4}, \beta_{i+1}, \alpha_{i+5})$

$C_{\beta_i} =_{\beta_{i+1}} C = K(\beta_{i+4}, \alpha_{i+4}, \alpha_i)$

P_{15}*is the group with cycle presentation given by atoms* :

$A_{P_{15}} := \{\alpha_i, \beta_i, \gamma_i, \delta_i, \epsilon_i \ i \in \mathbb{Z}_3\}$

and cycles :

$C_{\alpha_i} =_{\alpha_{i+1}} C = K(\delta_i, \beta_i, \epsilon_{i+1}, \gamma_{i+2})$

$C_{\beta_i} =_{\beta_{i+1}} C = K(\alpha_i, \delta_i, \gamma_i, \epsilon_i)$

$C_{\gamma_i} =_{\gamma_{i+1}} C = K(\alpha_{i+2}, \epsilon_i, \beta_i, \delta_{i+1})$

$C_{\delta_i} =_{\delta_{i+1}} C = K(\alpha_{i+1}, \gamma_i, \beta_i)$

$C_{\epsilon_i} =_{\epsilon_{i+1}} C = (\alpha_i, \beta_i, \gamma_{i+1})$

P_{19}*is the group with cycle presentation given by atoms* :

$A_{P_{19}} := \{\alpha_i \ i \in \mathbb{Z}_{12}, \ \beta_i \ i \in \mathbb{Z}_3, \ \gamma_i \ i \in \mathbb{Z}_4\}$

and cycles :

$C_{\alpha_i} =_{\alpha_{i+1}} C = K(\alpha_{i+4}, \gamma_i, \alpha_{i+9}, \beta_{i+1})$

$C_{\beta_i} =_{\beta_{i+1}} C = K(\alpha_i, \alpha_{i+9}, \alpha_{i+6}, \alpha_{i+3})$

$C_{\gamma_i} =_{\gamma_{i+1}} C = K(\alpha_{i+1}, \alpha_{i+5}, \alpha_{i+9})$

P_{24}*is the group with cycle presentation given by atoms* :

$A_{P_{24}} := \{\alpha_i, \beta_i, \gamma_i, \delta_i, \epsilon_i, \eta_i \ i \in \mathbb{Z}_4\}$

and cycles :

$C_{\alpha_i} =_{\alpha_{i+1}} = K(\beta_i, \gamma_i, \epsilon_i, \eta_i)$

$C_{\beta_i} =_{\beta_{i+1}} C = K(\alpha_{i+1}, \eta_{i+1}, \delta_i, \gamma_i)$

$C_{\gamma_i} =_{\gamma_{i+1}} C = K(\alpha_{i+1}, \beta_{i+1}, \delta_{i+1}, \epsilon_i)$

$C_{\delta_i} =_{\delta_{i+1}} C = K(\beta_{i+1}, \eta_{i+2}, \epsilon_{i+3}, \gamma_i)$

$C_{\epsilon_i} =_{\epsilon_{i+1}} C = K(\alpha_{i+1}, \gamma_{i+1}, \delta_{i+2}, \eta_i)$

$C_{\eta_i} =_{\eta_{i+1}} C = K(\alpha_{i+1}, \epsilon_{i+1}, \delta_{i+3}, \beta_i)$

Lemma 5.1.7 *Let* (G, Δ) *be a Bessis-Garside pair of rank* 3, *and consider the cycle presentation of* G *satisfying Condition 4.3.2 which exists as a result of Theorem 4.3.1. Suppose that* $l(a) \leq 4$ *for all* $a \in A$, *and that* $\tau_G = (4, 4, 4)$, *then* G *is isomorphic to one of* $\tilde{C}_2(k)$, P_{14}, P_{19}, *or* P_{24}.

Proof.

So taking the standard triple (a, b, c), $\Delta = abc$, we begin with a subset of A, $A_0 = \{a, b, c, d := b^{-1}ab, e := b^{-1}a^{-1}bab, f := c^{-1}bc, g := c^{-1}b^{-1}cbc, h := c^{-1}ac, i := c^{-1}a^{-1}cac\}$, and we know certain cycles on these atoms: ${}_cC = K(a, b, d, e)$, $C_a = K(b, c, f, g)$, and $K(a, c, h, i)$.

Now consider the triple (b, c, a) which implies $c \setminus d = f \setminus h$. ${}_{c\setminus d}C = K(b, c, f, g)$ so $c \setminus d$ is not in $A_0 - \{d, f, g, h\}$ as then $l(c \setminus d) \geq 5$. If $c \setminus d = f$ then we have the cycle $K(h, f, f)$ so $h = f$; if $c \setminus d = g$ then $c \setminus g = g$ whereas $c \setminus g = f$; and if $c \setminus d = h$ then we have the cycle $K(\ldots, d, c, h, \ldots) = K(a, c, h, i)$ so that $a = d$. So if $c \setminus d$ is in A_0 then it must be d. We suppose that this is the case and note the implied cycles $C_a =_d C = K(b, c, f, g)$, ${}_fC = K(a, c, h, i)$, $C_b = K(d, c)$, and $C_c = K(\ldots, h, f, d, \ldots)$.

From the triple (b, c, d) we have $d \setminus f = c \setminus e$. ${}_{c\setminus e}C = K(d, c)$ so $l(c \setminus e) = 2$ and $c \setminus e$ is not in $A_0 - \{b, e, h, i\}$. If $c \setminus e = b$ then $e \setminus b = c$ whereas $e \setminus b = a$; if $c \setminus e = e$ then we have cycle $K(h, f, d, e) = K(a, b, d, e)$ so that $b = f$; if $c \setminus e = h$ then we have cycle $K(\ldots, e, c, h, \ldots) = K(a, c, h, i)$ so that $a = e$; and if $c \setminus e = i$ then $h \setminus i = f$ whereas $h \setminus i = i$. So $c \setminus e$ is not in A_0 and we put $j := c \setminus e$ and $A_1 := A_0 \cup \{j\}$, and note cycles $C_c = K(h, f, d, j)$, $C_d = K(\ldots, e, c, j, \ldots)$, and $C_b =_j C = K(d, c)$.

From the triple (c, d, f) we have $j \setminus c = d \setminus g$. ${}_{j\setminus c}C = K(h, f, d, j)$ so $j \setminus c$ is not in $A_1 - \{e, g, i\}$ as then $l(j \setminus c) \geq 5$. If $j \setminus c = e$ then we have cycle $K(\ldots, g, d, e, \ldots) = K(a, b, d, e)$ so that $b = g$; if $j \setminus c = g$ then $c \setminus g = j$ whereas $c \setminus g = f$; and if $j \setminus c = i$ then $c \setminus i = j$ whereas $c \setminus i = h$. So $j \setminus c$ is not in A_1, and we put $k := j \setminus c$ and $A_2 := A_1 \cup \{k\}$, and note cycles $C_d = K(e, c, j, k)$, $C_f = K(\ldots, g, d, k, \ldots)$, $C_c =_k C = K(h, f, d, j)$, and $C_g =_c C = K(a, b, d, e)$.

Since $l(j) = 2$ and $h, k \in R(j)$, we have $C_j = K(h, k)$ and $C_d =_h C = K(e, c, j, k)$.

From (h, j, k) we have $h \setminus e = k \setminus f$. ${}_{h \setminus e}C = K(h, k)$, so $l(h \setminus e) = 2$ and $h \setminus e$ is not in $A_2 - \{b, i, j\}$. If $h \setminus e = b$ then $e \setminus b = h$ whereas $e \setminus b = a$; if $h \setminus e = i$ then we have cycle $K(\ldots, e, h, i, \ldots) = K(a, c, h, i)$ so that $e = c$; and if $h \setminus e = j$ then $h \setminus j = j$ whereas $h \setminus j = f$. So $h \setminus e$ is not in A_2, and we put $l := h \setminus e$ and $A_3 := A_2 \cup \{l\}$, and note cycles $C_k = K(\ldots, e, h, l, \ldots)$, $C_h = K(\ldots, f, k, l, \ldots)$, and $C_j =_l C = K(h, k)$.

From (f, h, k) we have $l \setminus h = k \setminus d$. ${}_{l \setminus h}C = K(\ldots, f, k, l, \ldots)$, so $l \setminus h$ is not in $A_3 - \{e, g, i, l\}$ as then $l(l \setminus h) \geq 5$. If $l \setminus h = e$ then $d \setminus e = k$ whereas $d \setminus e = e$; if $l \setminus h = g$ then we have cycle $C_f = K(g, d, k)$ and $l(f) = 3$; if $l \setminus h = i$ then $h \setminus i = l$ whereas $h \setminus i = i$; and if $l \setminus h = l$ we have cycle $K(e, h, l, l)$ so that $h = l$. So $l \setminus h$ is not in A_3, and we put $m := l \setminus h$ and $A_4 = A_3 \cup \{m\}$, and note cycles $C_f =_e C = K(g, d, k, m)$, $C_k = K(e, h, l, m)$, and $C_h =_m C = K(\ldots, f, k, l, \ldots)$.

From (e, h, k) we have $C_h =_m C = K(f, k, l, i)$ and $C_k =_i C = K(e, h, l, m)$. Since $l(l) = 2$ and $i, m \in R(l)$, we have $C_l = K(i, m)$. Likewise $l(b) = 2$ and $a, g \in L(b)$ so ${}_bC = K(a, g)$. From (h, i, k) we have cycle $C_i =_g C = K(\ldots, a, f, m, \ldots)$.

From (i, l, m) we have $i \setminus e = m \setminus f$. ${}_{i \setminus e}C = K(i, m)$, so $l(i \setminus e) = 2$ and $i \setminus e$ is not in $A_4 - \{b, j, l\}$. If $i \setminus e = b$ then $a \setminus b = f$ whereas $a \setminus b = b$; if $i \setminus e = j$ then $f \setminus j = m$ whereas $f \setminus j = d$; and if $i \setminus e = l$ then $m \setminus l = l$ whereas $m \setminus l = e$. So $i \setminus e$ is not in A_4, and we put $n := i \setminus e$ and $A_5 = A_4 \cup \{n\}$, and note cycle $C_i =_g C = K(a, f, m, n)$, $C_m =_a C = K(\ldots, e, i, n, \ldots)$, and $C_l =_n C = K(i, m)$.

From (f, i, m) we have cycle $C_m =_a C = K(e, i, n, g)$ and $C_n =_b C = K(a, g)$.

It is then clear that if $c \setminus d = d$ then $A = A_5$, and relabeling the atoms, $G = \tilde{C}_2(2)$.

If we return to the triple (a, b, c) and suppose that $c \setminus d \neq d$, and put instead $j := c \setminus d$ and $B_0 = A_0 \cup \{j\}$, then we have cycles $C_b = K(\ldots, d, c, j, \ldots)$, $C_c = K(\ldots, h, f, j, \ldots)$ and $C_a =_j C = K(b, c, f, g)$.

From (b, c, d) we have $c \setminus e = j \setminus f$. ${}_{c \setminus e}C = K(\ldots, d, c, j, \ldots)$ so $c \setminus e$ is not in $B_0 - \{e, f, g, h, i\}$ as then $l(c \setminus e) \geq 5$. If $c \setminus e = f$ then we have cycle $K(h, f, j, f)$ so that $h = j$; if $c \setminus e = g$ then $f \setminus g = j$ whereas $f \setminus g = g$; if $c \setminus e = h$ then we have cycle $K(\ldots, e, c, h, \ldots) = K(a, c, h, i)$ so that $a = e$; and if $c \setminus e = i$ then $c \setminus i = i$ whereas $c \setminus i = h$. So if $c \setminus e$ is in B_0 then it equals e. Suppose that this is the case then we have cycles $C_c = K(h, f, j, e)$, $C_d =_h C = K(e, c)$, and ${}_kC = K(\ldots, d, c, j, \ldots)$.

Since $l(h) = 2$ and $i, f \in R(h)$ we have $C_h = K(i, f)$. From (c, f, h) we have cycles $C_f = K(\ldots, g, j, i, \ldots)$, and ${}_iC = K(h, f, j, e)$.

From (f, h, i) we have $f \setminus a = i \setminus j$. ${}_{f \setminus a}C = K(i, f)$ so $l(f \setminus a) = 2$ and $f \setminus a$ is not in $B_0 - \{d, g, h\}$. If $f \setminus a = d$ then $a \setminus d = f$ whereas $a \setminus d = b$; if $f \setminus a = g$ then we have cycles $K(\ldots, a, f, g, \ldots) = K(b, c, f, g)$ so that $a = c$; and if $f \setminus a = h$ then $a \setminus h = f$ whereas $a \setminus h = c$. So $f \setminus a$ is not in $B0$, and we put $k := f \setminus a$ and $B_1 := B_0 \cup \{k\}$, and note cycles $C_i =_g C = K(\ldots, a, f, k, \ldots)$, $C_f = K(g, j, i, k)$, and ${}_kC = K(i, f)$.

From (f, i, j) we have $i \setminus e = k \setminus f$. ${}_{i \setminus e}C = K(g, j, i, k)$ so $i \setminus e$ is not in B_1 as then $l(i \setminus e) \geq 5$, so we put $l := i \setminus e$ and $B_2 := B_1 \cup \{l\}$, and note cycles $C_i =_g C = K(a, f, k, l)$, $C_j =_a C = K(\ldots, e, i, l, \ldots)$, and ${}_lC = K(g, j, i, k)$.

Since $l(k) = 2$ and $g, l \in R(k)$ we have $C_k = K(g, l)$.

From (g, k, l) we have $g \setminus a = l \setminus j$. ${}_{g \setminus a}C = K(g, l)$ so $l(g \setminus a) = 2$, and $g \setminus a$ is not in $B_2 - \{d, h, k\}$. If $g \setminus a = d$ then $a \setminus d = g$ whereas $a \setminus d = b$; if $g \setminus a = h$ then $a \setminus h = g$ whereas $a \setminus h = c$; and if $g \setminus a = k$ then $j \setminus k = l$ whereas $j \setminus k = i$. Therefore $g \setminus a$ is not in B_2, so we put $m := g \setminus a$ and $B_3 = B_2 \cup \{m\}$, and note cycles $C_l =_b C = K(\ldots, a, g, m, \ldots)$, $C_g = K(\ldots, j, l, m, \ldots)$, and $C_k =_m C = K(g, l)$.

From (a, g, l) we have $C_g = K(j, l, m, b)$.

From (g, j, l) we have $l \setminus i = m \setminus g$. ${}_{l \setminus i}C = K(j, l, m, b)$ so $l \setminus i$ is not in $B_3 - \{d\}$ as then $l(l \setminus i) \geq 5$. If $l \setminus i = d$ then $e \setminus d = i$ whereas $e \setminus d = a$. Therefore $l \setminus i$ is not in B_3, so we put $n := l \setminus i$ and $B_4 = B_3 \cup \{n\}$, and note cycle $C_j =_a C = K(e, i, l, n)$, $C_l =_b C = K(a, g, m, n)$, $C_g =_n C = K(b, j, l, m)$, and $C_n =_c C = K(a, b, d, e)$.

Since $l(m) = 2$ and $b, n \in R(m)$, we have $C_m = K(b, n)$.

Finally from (b, f, l) we have cycles $C_e =_b C = K(d, c, j, n)$ and $C_m =_d C = K(b, n)$.

So in case 6, if $c \setminus d \neq d$ and $c \setminus e = e$, then $A = B_4$, and we can easily relabel the atoms to see that $G = \tilde{C}_2(2)$.

Returning to the triple (b, c, d), we now suppose $c \setminus e \neq e$, and put $k := c \setminus e$ and $D_0 := B_0 \cup \{k\}$. We then have cycles $C_c = K(h, f, j, k)$, $C_d =_h C = K(\ldots, e, c, k, \ldots)$, and $C_b =_k C = K(\ldots, d, c, j, \ldots)$.

From (c, f, h) we have $j \setminus g = f \setminus i$. ${}_{j \setminus g}C = K(h, f, j, k)$ so $j \setminus g$ is not in $D_0 - \{g, i\}$, as then $l(j \setminus g) \geq 5$. If $j \setminus g = g$ then we have cycle $K(\ldots, i, f, g, \ldots) = K(b, c, f, g)$ so that $c = i$; and if $j \setminus g = i$ then we have $C_h = K(i, f)$ so $l(h) = 2$. Therefore $j \setminus g$ is not in D_0, so we put $l := j \setminus g$ and $D_1 := D_0 \cup \{l\}$, and note cycles $C_f = K(\ldots, g, j, l, \ldots)$, $C_h =_a C = K(\ldots, i, f, l, \ldots)$, and $C_c =_l C = K(h, f, j, k)$.

From (f, h, i) we have $f \setminus a = l \setminus j$. ${}_{f \setminus a}C = K(\ldots, i, f, l, \ldots)$, so $f \setminus a$ is not in $D_1 - \{a, d, e, g, i\}$ as then $l(f \setminus a) \geq 5$. If $f \setminus a = d$ then $f \setminus a = d$ then $a \setminus d = f$ whereas $a \setminus d = b$; if $f \setminus a = e$ then $a \setminus e = f$ whereas $a \setminus e = b$; if $f \setminus a = g$ then we have cycle $K(g, j, l)$ and $l(f) = 3$; and if $f \setminus a = i$ then $a \setminus i = f$ whereas $a \setminus i = c$. So if $f \setminus a \in D_1$, then $f \setminus a = a$. Suppose that this is the case, then we have cycles $C_i =_g C = K(a, f)$, $C_f =_b C = K(g, j, l, a)$, and $C_h =_a C = K(\ldots, i, f, l, \ldots)$.

Since $l(g) = 2$ and $b, j \in R(g)$, we have $C_g = K(b, j)$.

From (b, g, j) we have $b \setminus l = j \setminus c$. ${}_{b \setminus l}C = K(b, j)$, so $l(b \setminus l) = 2$, and $b \setminus l$ is not in $D_1 - \{g, i\}$. If $b \setminus l = g$ then $b \setminus g = g$ whereas $b \setminus g = c$; and if $b \setminus l = i$ then $c \setminus i = j$ whereas $c \setminus i = c$. So $b \setminus l$ is not in D_1, so we put $m := b \setminus l$ and $D_2 = D_1 \cup \{m\}$, and note cycles $C_b =_k C = K(d, c, j, m)$, $C_j =_d C = K(\ldots, l, b, m, \ldots)$, $C_g =_m C = K(b, j)$, and $C_l =_c C = K(a, b, d, e)$.

From (b, c, j) we have $C_j =_d C = K(l, b, m, k)$. Since $l(m) = 2$ and $d, k \in R(m)$ we have $C_m = K(d, k)$.

From (b, d, k) we have $k \setminus e = d \setminus l$. ${}_{k \setminus e}C = K(d, k)$, so $l(k \setminus e) = 2$, and $k \setminus e$ is not in $D_2 - \{g, i, m\}$. If $k \setminus e = g$ then $c \setminus g = k$ whereas $c \setminus g = f$; if $k \setminus e = i$ then $c \setminus i = i$ whereas $c \setminus i = h$; and if $k \setminus e = m$ then $d \setminus m = m$ whereas $d \setminus m = c$. So $k \setminus e$ is not in D_2, so put $n := k \setminus e$ and $D_3 = D_2 \cup \{n\}$, and note cycles $C_d =_h C = K(e, c, k, n)$,

$C_k =_e C = K(\ldots, l, d, n, \ldots)$, and $C_m =_n C = K(d, k)$.

From (d, j, k) we have $C_k =_e C = K(h, l, d, n)$. Since $l(n) = 2$ and $e, h \in R(n)$ we have $C_n = K(e, h)$. From (d, h, j) we have $C_h =_a C = K(i, f, l, e)$, and $C_n =_i C = K(e, h)$.

So in case 6, if $c \setminus d \neq d$, $c \setminus e \neq e$, and $f \setminus a = a$ then $A = D_3$, and we can relabel atoms to see that $G = \tilde{C}_2(2)$.

Returning to the triple (f, h, i), we now suppose that $f \setminus a \neq a$, and put $m := f \setminus a$ and $E_0 = D_1 \cup \{m\}$. We then have cycles $C_i = K(\ldots, a, f, m, \ldots)$, $C_f = K(g, j, l, m)$, and $C_h =_m C = K(\ldots, i, f, l, \ldots)$.

From (f, g, j) we have $j \setminus b = l \setminus k$. ${}_{j\setminus b}C = K(g, j, l, m)$, so $j \setminus b$ is not in E_0 as then $l(j \setminus b) \geq 5$. So we put $n := j \setminus b$ and $E_1 = E_0 \cup \{n\}$, and note cycles $C_f =_n C = K(g, j, l, m)$, $C_j = K(\ldots, k, l, n, \ldots)$, and $C_g = K(\ldots, b, j, n \ldots)$.

From (b, g, j) we have $j \setminus c = n \setminus l$. ${}_{j\setminus c}C = K(\ldots, b, j, n, \ldots)$, so $j \setminus c$ is not in $E_1 - \{d, e, i\}$ as then $l(j \setminus c) \geq 5$. If $j \setminus c = e$ then $d \setminus e = c$ whereas $d \setminus e = e$; and if $j \setminus c = i$ then $c \setminus i = j$ whereas $c \setminus i = h$. So if $j \setminus c \in E_1$ then $j \setminus c = d$. Suppose that this is the case and note cycles $C_b =_k C = K(d, c, j)$, $C_j = K(k, l, n, d)$, and $C_g =_d C = K(\ldots, b, j, n, \ldots)$.

From (j, k, l) we have $l \setminus h = n \setminus m$. ${}_{l\setminus h}C = K(k, l, n, d)$, so $l \setminus h$ is not in $E_1 - \{e\}$ as then $l(l \setminus h) \geq 5$. Suppose that $l \setminus h = e$ then we have cycles $C_l = K(\ldots, m, n, e, \ldots)$, $C_k = K(h, l, e)$, $C_j =_e C = K(k, l, n, d)$, and $C_n =_c C = K(a, b, d, e)$.

From (d, j, k) we get $C_d =_h C = K(e, c, k)$, $C_g =_d C = K(b, j, n)$, $C_h =_m C = K(i, f, l)$, $C_i =_g C = K(a, f, m)$, and $C_k =_i C = K(h, l, e)$. From (h, k, l) we have $C_l =_a C = K(m, n, e, i)$. From (f, j, m) we have $C_m =_b C = K(a, g, n)$.

So in case 6, if $c \setminus d \neq d$, $c \setminus e \neq e$, $f \setminus a \neq a$, $j \setminus c = d$, and $l \setminus h = e$, then $A = E_1$, and we can relabel atoms to see that $G = P_{14}$.

Returning to the triple (j, k, l), we suppose that $l \setminus h \neq e$, and put $p := l \setminus h$ and $F_0 = E_1 \cup \{p\}$. We then have cycles $C_l = K(\ldots, m, n, p, \ldots)$, $C_k = K(h, l, p)$, and $C_j =_p C = K(k, l, n, d)$.

From (h, k, l) we have $l \setminus f = p \setminus n$. ${}_{l\setminus f}C = K(h, l, p)$ so $l(l \setminus f) = 3$ and $l \setminus f$ is not in $\{a, c, e, f, j, l, n, p\}$; and $l \setminus f$ is not in $\{b, d, g, h, k, m\}$ as then $l(l \setminus f) \geq 5$. If $l \setminus f = i$ then $C_h = K(i, f, l)$, so ${}_hC = K(e, c, k)$ and therefore $e \in R(k) = \{h, l, p\}$. So $l \setminus f$ is not in F_0, so put $q := l \setminus f$ and $F_1 = F_0 \cup \{q\}$, and note cycles $C_h =_m C = K(i, f, l, q)$, $C_l = K(m, n, p, q)$, and $C_k =_q C = K(h, l, p)$.

From (l, m, n) we have $n \setminus g = p \setminus d$. ${}_{n\setminus g}C = K(m, n, p, q)$, so $n \setminus g$ is not in F_1 as then $l(n \setminus g) \geq 5$, so we put $r := n \setminus g$ and $F_2 = F_1 \cup \{r\}$, and note cycles $C_l =_r C = K(m, n, p, q)$, $C_m = K(\ldots, g, n, r, \ldots)$, and $C_n = K(\ldots, d, p, r, \ldots)$.

From (g, m, n) we have $n \setminus j = r \setminus p$. ${}_{n\setminus j}C = K(\ldots, g, n, r, \ldots)$ so $l(n \setminus j) = 4$ and $n \setminus j$ is not in $\{b, k, q\}$; and $n \setminus j$ is not in $F_2 - \{b, e, k, q\}$ as then $l(n \setminus j) \geq 5$. If $n \setminus j = e$ then $b \setminus e = j$ whereas $b \setminus e = d$. So $n \setminus j$ is not in F_2, so we put $s := n \setminus j$ and $F_3 := F_2 \cup \{s\}$, and note cycles $C_g =_d C = K(b, j, n, s)$, $C_n = K(d, p, r, s)$, and $C_m =_s C = K(\ldots, g, n, r, \ldots)$.

From (d, j, l) we have $C_d =_h C = K(e, c, k, p)$, and $C_n =_e C = K(d, p, r, s)$. From (d, n, p) we have $C_p = K(\ldots, q, r, e, \ldots)$. From (l, n, q) we have $C_q = K(i, m, r)$, and $C_p =_i C = K(\ldots, q, r, e, \ldots)$. From (c, d, p) we have $C_p =_i C = K(q, r, e, h)$.

From (i,m,q) we have $m \setminus f = r \setminus n$. ${}_{m\setminus f}C = K(i,m,r)$ so $l(m \setminus f) = 3$ and $m \setminus f$ is not in $F_3 - \{b,k,q\}$; and $m \setminus f$ is not in $\{b,k,q\}$ as then $l(m \setminus f) \geq 5$. So put $t := m \setminus f$ and $F_4 := F_3 \cup \{t\}$, and note cycles $C_i =_g C = K(a,f,m,t)$, $C_m =_s C = K(g,n,r,t)$, $C_q =_t C = K(i,m,r)$.

From (m,n,r) we have $C_r =_a C = K(\ldots,i,t,s,\ldots)$. From (n,p,r) we have $C_r =_a C = K(i,t,s,e)$. From (m,r,t) we have $C_t =_b C = K(a,g,s)$.

So in case 6, if $c \setminus d \neq d$, $c \setminus e \neq e$, $f \setminus a \neq a$, $j \setminus c = d$, and $l \setminus h \neq e$, then $A = F_4$, and we can relabel atoms to see that $G = P_{19}$.

Returning to the triple (b,g,j) we suppose that $j \setminus c \neq d$, and put $p := j \setminus c$ and $G_0 = E_1 \cup \{$. We then have cycles $C_b =_k C = K(d,c,j,p)$, $C_j = K(k,l,n,p)$, and $C_g =_p C = K(\ldots,b,j,n,\ldots)$.

From (j,k,l) we have $l \setminus h = n \setminus m$. ${}_{l\setminus h}C = K(k,l,n,p)$, so $l \setminus h$ is not in G_0 as then $l(l \setminus h) \geq 5$. So we put $q := l \setminus h$ and $G_1 := G_0 \cup \{q\}$, and note cycles $C_l = K(\ldots,m,n,q,\ldots)$, $C_k = K(\ldots,h,l,q,\ldots)$, and $C_j =_q C = K(k,l,n,p)$.

From (h,k,l) we have $l \setminus f = q \setminus n$. ${}_{l\setminus f}C = K(\ldots,h,l,q,\ldots)$, so $l \setminus f$ is not in $G_1 - \{e,i,m\}$ as then $l(l \setminus f) \geq 5$. If $l \setminus f = e$ then we have $K(m,n,q,e)$, and therefore $l(e) \geq 5$; if $l \setminus f = m$ then we have $C_l = K(m,n,q)$ so that $l(l) = 3$ whereas $l(l) = 4$. So if $l \setminus f \in G_1$, then $l \setminus f = i$. We suppose that this is the case, and note that this gives us cycles $C_l = K(m,n,q,i)$, $C_h =_m C = K(i,f,l)$, $C_d =_h C = K(e,c,k)$, and $C_k =_i C = K(\ldots,h,l,q,\ldots)$.

From (i,l,n) we have $m \setminus f = q \setminus p$. ${}_{m\setminus f}C = K(m,n,q,i)$, so $m \setminus f$ is not in G_1 as then $l(m \setminus f) \geq 5$, so we put $r := m \setminus f$ and $G_2 := G_1 \cup \{r\}$, and note cycles $C_n = K(\ldots,p,q,r,\ldots)$, $C_i =_g C = K(a,f,m,r)$, $C_l =_r C = K(m,n,q,i)$.

From (l,m,n) we have $C_m = K(g,n,r)$.

From (g,l,n) we have $n \setminus g = r \setminus q$. ${}_{n\setminus g}C = K(g,n,r)$ so $l(n \setminus g) = 3$ and $n \setminus g$ is not in $G_2 - \{d,h,m\}$; and $n \setminus g$ is not in $\{d,h,m\}$ as then $l(n \setminus g) \geq 5$. So we put $s := n \setminus g$ and $G_3 = G_2 \cup \{s\}$, and note cycles $C_n = K(p,q,r,s)$, $C_g =_p C = K(b,j,n,s)$, and $C_m =_s C = K(g,n,r)$.

From (n,p,q) we have $q \setminus k = r \setminus i$. ${}_{q\setminus k}C = K(p,q,r,s)$ so $q \setminus k$ is not in G_3 as then $l(q \setminus k) \geq 5$, so we put $t := q \setminus k$ and $G_4 = G_3 \cup \{t\}$, and note cycles $C_n =_t C = K(p,q,r,s)$, $C_q = K(\ldots,i,r,t,\ldots)$, and $C_p = K(\ldots,k,q,t,\ldots)$.

From (n,q,r) and (i,q,r) we have $C_r =_b C = K(g,s,t,a)$ and $C_q =_a C = K(\ldots,i,r,t,\ldots)$. From (a,r,s) we have $C_s =_d C = K(p,t,b)$, and $C_t =_c C = K(a,b,d,e)$. From (p,s,t) we have $C_p =_e C = K(k,q,t,d)$. From (d,k,p) we have $C_k =_i C = K(h,l,q,e)$. From (k,n,q) we have $C_q =_a C = K(i,r,t,e)$.

So in case 6, if $c \setminus d \neq d$, $c \setminus e \neq e$, $f \setminus a \neq a$, $j \setminus c \neq d$, $l \setminus h \neq e$, and $l \setminus f = i$ then $A = G_4$, and we can relabel atoms to see that $G = P_{19}$.

Returning to the triple (h,k,l) we suppose that $l \setminus f \neq i$, and put $r := l \setminus f$ and $H_0 = G_1 \cup \{r\}$. We then have cycles $C_h =_m C = K(i,f,l,r)$, $C_l = K(m,n,q,r)$, and $C_k =_r C = K(\ldots,h,l,q,\ldots)$.

From (l,m,n) we have $n\backslash g = q\backslash p$. ${}_{n\backslash g}C = K(m,n,q,r)$, so $n\backslash g$ is not in H_0 as then $l(n\backslash g) \geq 5$, so we put $s := n\backslash g$ and $H_1 = H_0 \cup \{s\}$, and note cycles $C_n = K(\ldots,p,q,s,\ldots)$, $C_m = K(\ldots,g,n,s,\ldots)$, and $C_l =_s C = K(m,n,q,r)$.

From (g,m,n) we have $n \setminus j = s \setminus q$. ${}_{n\setminus j}C = K(\ldots,g,n,s,\ldots)$, so $n \setminus j$ is not in $H_1 - \{b,e\}$ as then $l(n \setminus j) \geq 5$. If $n \setminus j = e$ then $b \setminus e = j$ whereas $b \setminus e = d$. So if $n \setminus j \in H_1$ then $n \setminus j = b$. We suppose that this is the case and note that we then have cycles $C_g =_p C = K(b,j,n)$, $C_i =_g C = K(a,f,m)$, $C_n =_d C = K(p,q,s,b)$, and $C_m =_b C = K(\ldots,g,n,s,\ldots)$.

From (b,d,p) we have $C_p = K(q,d,k)$.

From (d,p,q) we have $k \setminus c = d \setminus s$. ${}_{k\setminus c}C = K(d,k,q)$, so $l(k \setminus c) = 3$ and $k \setminus c$ is not in $H_1 - \{g,i,p\}$; and $k \setminus c$ is not in $\{g,i,p\}$ as then $l(k \setminus c) \geq 5$. So we put $t := k \setminus c$ and $H_2 := H_1 \cup \{t\}$, and note cycles $C_d =_h C = K(e,c,k,t)$, $C_q =_e C = K(\ldots,s,d,t,\ldots)$, $C_s =_c C = K(a,b,d,e)$, and $C_p =_t C = K(d,k,q)$.

From (n,p,q) we have $C_q =_e C = K(r,s,d,t)$. From (d,q,r) we have $C_r =_a C = K(\ldots,m,s,e,\ldots)$ From (m,q,s) we have $C_m =_b C = K(a,g,n,s)$. From (k,p,q) we have $C_k =_r C = K(h,l,q,t)$. From (b,e,t) we have $C_t =_i C = K(e,h,r)$. From (d,e,r) we have $C_r =_a C = K(m,s,e,i)$.

So in case 6, if $c \setminus d \neq d$, $c \setminus e \neq e$, $f \setminus a \neq a$, $j \setminus c \neq d$, $l \setminus h \neq e$, $l \setminus f \neq i$, and $n \setminus j = b$ then $A = H_2$, and we can relabel atoms to see that $G = P_{19}$.

Returning to the triple (g,m,n) we suppose that $n \setminus j \neq b$, and put $t := n \setminus j$ and $I_0 = H_1 \cup \{t\}$. We then have cycles $C_g =_p C = K(b,j,n,t)$, $C_n = K(p,q,s,t)$, and $C_m =_t C = K(\ldots,g,n,s,\ldots)$.

From (n,p,q) we have $q \setminus k = s \setminus r$. ${}_{q\setminus k}C = K(p,q,s,t)$ so $q \setminus k$ is not in I_0 as then $l(q \setminus k) \geq 5$, so put $u := q \setminus k$ and $I_1 := I_0 \cup \{u\}$, and note cycles $C_n =_u C = K(p,q,s,t)$, $C_p = K(\ldots,k,q,u,\ldots)$, and $C_q = K(\ldots,r,s,u,\ldots)$.

From (n,q,t) we have $C_t =_d C = K(\ldots,b,p,u,\ldots)$. From (b,p,t) we have $C_p = K(k,q,u,d)$.

From (k,p,q) we have $q \setminus l = u \setminus s$. ${}_{q\setminus l}C = K(k,q,u,d)$ so $q \setminus l$ is not in I_1 as then $l(q \setminus l) \geq 5$, so put $v := q \setminus l$ and $I_2 := I_1 \cup \{v\}$, and note cycles $C_k =_r C = K(h,l,q,v)$, $C_p =_v C = K(k,q,u,d)$, and $C_q = K(r,s,u,v)$.

From (d,p,q) we have $C_d =_h C = K(e,c,k,v)$.

From (q,s,v) we have $r \setminus h = u \setminus t$. ${}_{r\setminus h}C = K(r,s,u,v)$ so $r \setminus h$ is not in I_2 as then $l(r \setminus h) \geq 5$, so put $w := r \setminus h$ and $I_3 := I_2 \cup \{w\}$, and note cycles $C_v =_i C = K(\ldots,h,r,w,\ldots)$, $C_s = K(\ldots,t,u,w,\ldots)$, and $C_q =_w C = K(r,s,u,v)$.

From (h,r,v), and (f,h,r) we have $C_r = K(s,w,i,m)$.

From (i,r,s) we have $m \setminus f = w \setminus u$. ${}_mC = K(s,w,i,m)$ so $m \setminus f$ is not in I_3 as then $l(m \setminus f) \geq 5$, so put $x := m \setminus f$ and $I_4 := I_3 \cup \{x\}$, and note cycles $C_i =_g C = K(a,f,m,x)$, $C_s = K(t,u,w,x)$, and $C_r =_x C = K(s,w,i,m)$.

From (m,q,s) we have $C_m =_t C = K(g,n,s,x)$. From (f,g,x) we have $C_x =_b C = K(\ldots,a,g,t,\ldots)$.

From (s, t, u) we have $u \setminus p = w \setminus v$. ${}_{u\setminus p}C = K(t, u, w, x)$ so $u \setminus p$ is not in I_4 as then $l(u \setminus p) \geq 5$, so put $y := u \setminus p$ and $I_5 := I_4 \cup \{y\}$, and note cycles $C_u = K(\ldots, v, w, y, \ldots)$, $C_t =_d C = K(b, p, u, y)$, and $C_s =_y C = K(t, u, w, x)$.

From (b, t, u) we have $C_u =_e C = K(v, w, y, d)$, and $C_y =_c C = K(a, b, d, e)$. From (d, q, v) we have $C_v =_i C = K(h, r, w, e)$. From (u, v, w) we have $C_w =_a C = K(\ldots, x, y, e, \ldots)$. From (r, v, w) we have $C_w =_a C = K(x, y, e, i)$. From (s, u, x) we have $C_x =_b C = K(a, g, t, y)$.

So in case 6, if $c \setminus d \neq d$, $c \setminus e \neq e$, $f \setminus a \neq a$, $j \setminus c \neq d$, $l \setminus h \neq e$, $l \setminus f \neq i$, and $n \setminus j \neq b$ then $A = I_5$, and we can relabel atoms to see that $G = P_{24}$.

This completes the proof.

■

Lemma 5.1.8 *Let (G, Δ) be a Bessis-Garside pair of rank 3, and consider the cycle presentation of G satisfying Condition 4.3.2 which exists as a result of Theorem 4.3.1. Suppose that $l(a) \leq 4$ for all $a \in A$, and that $\tau_G = (4, 4, 3)$, then G is isomorphic to one of $\tilde{C_2}(1)$, B_2, P_{14}, P_{15}, or P_{19}.*

Proof. Suppose we have standard triple (a, b, c), and associated cycles ${}_cC = K(a, b, d, e)$, $C_a = K(b, c, f, g)$, and $K(a, c, h)$. From the cube condition on (c, b, a) we can choose to assume that $c \setminus d = c$. We then consider the cube conditions on (d, c, a), (h, f, c), (g, d, b), (c, j, d), and (j, h, d); constructing along the way atoms $\{i := c \setminus e, j := f \setminus a, k := j \setminus f\}$. Then the set of atoms of G is $\{a, b, c, d, e, f, g, h, i, j, k\}$ which satisfy cycles:

$$\begin{aligned}
C_a &=_d C = K(b, c, f, g) \\
C_d &=_h C = K(e, c, i, j) \\
C_h &=_g C = K(a, f, j, k) \\
C_g &=_c C = K(a, b, d, e) \\
C_c &=_j C = K(h, f, d, i) \\
C_j &=_a C = K(g, e, h, k) \\
C_e &=_f C = K(a, c, h) \\
C_f &=_e C = K(g, d, j) \\
C_b &=_i C = K(d, c) \\
C_i &=_k C = K(h, j) \\
C_k &=_b C = K(a, g)
\end{aligned}$$

and relabelling $(a_0, a_1, c_0^0, c_1^0, c_2^0, b_0, b_1, b_2, c_0^1, c_1^1, c_2^1) := (e, f, a, c, h, b, i, k, d, j, b)$ we can see that G is equal to $\tilde{C_2}(1)$.

Returning to the triple (c, b, a) we assume $i := c \setminus d \neq c$. Considering the cube condition on (d, c, a) we can choose to assume that $c \setminus e = c$. Then from (h, f, c) we have that the set of atoms of G is $\{a, b, c, d, e, f, g, h\}$ which satisfy cycles:

$$\begin{aligned}
C_a &=_i C = K(b, c, f, g) \\
C_i &=_c C = K(a, b, d, e) \\
C_c &=_a C = K(h, f, i, e) \\
C_b &=_e C = K(d, c, i) \\
C_e &=_f C = K(a, c, h) \\
C_f &=_b C = K(g, i, a) \\
C_d &=_h C = K(e, c) \\
C_h &=_g C = K(a, f) \\
C_g &=_d C = K(b, i)
\end{aligned}$$

and relabelling $(\alpha_0, \alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2, \gamma_0, \gamma_1, \gamma_2) := (a, i, c, e, f, b, d, h, g)$ we can see that G is equal to B_2 (from example 4.6.2).

Returning to the triple (d, c, a), we assume $j := c \backslash e \neq c$. Considering triples (h, f, c), and (i, g, f) we construct distinct atoms $\{k : f \setminus a, l := i \setminus b\}$. Considering the cube condition on (b, i, g) we may assume that $i \setminus c = d$. We consider the cube condition for triples (d, j, i), (k, j, i), (k, h, i), (g, l, k), (m, l, k), (d, m, l), and (h, n, m); constructing along the way atoms $\{m : j \setminus c, n := k \setminus f, p := l \setminus i\}$. Then the atoms of G are $\{a, b, c, d, e, f, g, h, i, j, k, l, m, n, p\}$ which satisfy cycles:

$$C_a =_i C = K(b,c,f,g)$$
$$C_i =_m C = K(j,k,l,d)$$
$$C_m =_a C = K(h,n,p,e)$$
$$C_c =_k C = K(h,f,i,j)$$
$$C_k =_p C = K(g,l,m,n)$$
$$C_p =_c C = K(a,b,d,e)$$
$$C_d =_h C = K(e,c,j,m)$$
$$C_h =_g C = K(a,f,k,n)$$
$$C_g =_d C = K(b,i,l,p)$$
$$C_b =_j C = K(d,c,i)$$
$$C_j =_n C = K(h,k,m)$$
$$C_n =_b C = K(a,g,p)$$
$$C_e =_f C = K(a,c,h)$$
$$C_f =_l C = K(g,i,k)$$
$$C_l =_e C = K(d,m,p)$$

and relabelling the atoms we can see that G is P_{15}

Returning to the triple (b,i,g), we assume $m := i \setminus c \neq d$. Considering the triple (k,j,i), we construct the distinct atom $n : k \setminus h$. Considering the cube condition on (g,l,k) we may assume that $l \backslash i = b$. We consider the cube condition for triples (f,k,h), and (b,l,m) to show that the atoms of G are $\{a,b,c,d,e,f,g,h,i,j,k,l,m,n\}$ which satisfy cycles:

$$C_a =_i C = K(b, c, f, g)$$
$$C_i =_n C = K(j, k, l, m)$$
$$C_n =_c C = K(a, b, d, e)$$
$$C_c =_k C = K(h, f, i, j)$$
$$C_k =_b C = K(g, l, n, a)$$
$$C_b =_j C = K(d, c, i, m)$$
$$C_j =_a C = K(h, k, n, e)$$
$$C_d =_h C = K(e, c, j)$$
$$C_h =_g C = K(a, f, k)$$
$$C_g =_m C = K(b, i, l)$$
$$C_m =_e C = K(j, n, d)$$
$$C_e =_f C = K(a, c, h)$$
$$C_f =_l C = K(g, i, k)$$
$$C_l =_d C = K(m, n, b)$$

and relabelling the atoms we can see that G is P_{14}

Returning to the cycle (g, l, k) we assume that $p := l \setminus i \neq b$. Considering the triples of atoms (h, k, j), (n, m, l), (j, n, m), (q, p, n), (m, p, b), (r, p, n), (d, j, i), (d, r, s), (q, k, l), and (s, c, d); constructing along the way distinct atoms . We can then see that the atoms of G are $\{a, b, c, d, e, f, g, h, i, j, k, l, m, n, p, q, r, s, t\}$ which satisfy cycles:

$$C_a =_i C = K(b,c,f,g)$$
$$C_i =_n C = K(j,k,l,m)$$
$$C_n =_t C = K(q,p,r,s)$$
$$C_t =_c C = K(a,b,d,e)$$
$$C_c =_k C = K(h,f,i,j)$$
$$C_k =_p C = K(g,l,n,q)$$
$$C_p =_d C = K(m,r,t,b)$$
$$C_d =_h C = K(e,c,j,s)$$
$$C_h =_g C = K(a,f,k,q)$$
$$C_g =_m C = K(b,i,l,p)$$
$$C_m =_s C = K(j,n,r,d)$$
$$C_s =_a C = K(q,t,e,h)$$
$$C_b =_j C = K(d,c,i,m)$$
$$C_j =_q C = K(h,k,n,s)$$
$$C_q =_b C = K(g,p,t,a)$$
$$C_e =_f C = K(a,c,h)$$
$$C_f =_l C = K(g,i,k)$$
$$C_l =_r C = K(m,n,p)$$
$$C_r =_e C = K(d,s,t)$$

and relabelling the atoms we can see that G is P_{19}

This completes the proof ■

Lemma 5.1.9 *Let (G,Δ) be a Bessis-Garside pair of rank 3, and consider the cycle presentation of G satisfying Condition 4.3.2 which exists as a result of Theorem 4.3.1. Suppose that $l(a) \leq 4$ for all $a \in A$, and that $\tau_G = (4,3,3)$, then G is isomorphic to either $\tilde{A}_2(k)$ or P_{14}.*

Proof. Suppose we have standard triple (a,b,c), and associated cycles ${}_cC = K(a,b,d,e)$, $C_a = K(b,c,g)$, and $K(a,c,f)$. Much as in the proof of theorem 4.2.7 we use the fact that G is Garside, and therefore that the cube condition holds for all triples of atoms to construct cycles and atoms of G, until we have a cycle presentation satisfying condition 4.3.2. We consider the cube conditions on the triples of atoms (c,b,a), (g,f,c), and (d,c,a); constructing along the way atoms $\{h := c \setminus d, i := g \setminus a, j := c \setminus e\}$ distinct from one another. From the cube condition on (b,h,g) we can choose to assume that $h \setminus c = d$. We then consider the cube condition on (j,i,h) and we find that the set of atoms of G is $\{a,b,c,d,e,f,g,,h,i,j\}$ which satisfy cycles:

$$C_c =_i C = K(f, g, h, j)$$
$$C_i =_c C = K(a, b, d, e)$$
$$C_a =_h C = K(b, c, g)$$
$$C_h =_e C = K(j, i, d)$$
$$C_e =_g C = K(a, c, f)$$
$$C_g =_d C = K(b, k, i)$$
$$C_d =_f C = K(e, c, j)$$
$$C_f =_b C = K(a, g, i)$$
$$C_b =_j C = K(d, c, h)$$
$$C_j =_a C = K(f, i, e)$$

and relabelling $(a_0, a_1, c_0^0, c_1^0, c_2^0, c_3^0, c_0^1, c_1^1, c_2^1, c_3^1) := (c, i, f, g, h, j, a, b, d, e)$ we can see that G is equal to $\tilde{A_2}(4)$.

Returning to the triple (b, h, g) we assume $k := h \setminus c \neq d$. Consider the cube conditions on the triples (j, i, h), (f, i, j), (b, k, i), (d, j, k), (n, j, i), and (l, i, b); and construct in the process atoms $\{l := i \setminus f, m := i \setminus g, n := j \setminus c\}$ distinct from one another. Then the set of atoms of G is $\{a, b, c, d, e, f, g, h, i, j, k, l, m, n\}$ which satisfy cycles:

$$C_c =_i C = K(f, g, h, j)$$
$$C_i =_d C = K(b, k, l, m)$$
$$C_d =_f C = K(e, c, j, n)$$
$$C_f =_b C = K(a, g, i, m)$$
$$C_b =_j C = K(d, c, h, k)$$
$$C_j =_m C = K(f, i, l, n)$$
$$C_m =_c C = K(a, b, d, e)$$
$$C_a =_h C = K(b, c, g)$$
$$C_h =_l C = K(j, i, k)$$
$$C_l =_e C = K(m, n, d)$$
$$C_e =_g C = K(a, c, f)$$
$$C_g =_k C = K(b, h, i)$$
$$C_k =_n C = K(j, l, d)$$
$$C_n =_a C = K(e, f, m)$$

and relabelling $(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6) := (c, i, d, f, b, j, m, a, h, l, e, g, k, n)$ we can see that G is equal to P_{14}. This completes the proof ■

Theorem 5.1.10 *Let (G, Δ) be a Bessis-Garside pair of rank 3, and consider the cycle presentation of G satisfying Condition 4.3.2 which exists as a result of Theorem 4.3.1. Suppose that $l(a) \leq 4$ for all $a \in A$, then G is isomorphic to one of the following: $I_2(2) \times \mathbb{Z}$, $I_2(3) \times \mathbb{Z}$, $I_2(4) \times \mathbb{Z}$, the spherical Artin group B_2, a quotient of an affine Artin group: $\tilde{A}_2(2)$, $\tilde{A}_2(3)$, $\tilde{A}_2(4)$, $\tilde{C}_2(3)$, $\tilde{C}_2(4)$, or one of the exceptional cases P_{14}, P_{15}, P_{19} or P_{24}.*

Proof.

Recall the definition of τ_G. By lemmas 5.1.1, 5.1.5, 5.1.7, 5.1.8, reflem:433, and theorem 4.2.7; and lemma 5.1.4, we need only consider the following cases:

1. $\tau = (4, 2, 2)$,
2. $\tau = (4, 3, 2)\sigma$ for $\sigma \in \Sigma_3$,

For the first of these we choose standard triple (a, b, c), and we then have cycles $K(a, c)$, $K(b, c)$, and $C_c =_c C = K(a, b, d, e)$ for some atoms d, e. We have therefore cycles $K(d, c)$ and $K(e, c)$. So the atoms of G are $\{a, b, d, e, c\}$, and G is $I_2(4) \times \mathbb{Z}$.

For the second case we can use lemma 5.1.2 with $h = \sigma_2$ to reduce to the case when $\tau_G = (4, 3, 2)$. We choose standard triple (a, b, c), so that we have cycles $K(a, b, d, e)$, $K(b, c, f)$, and $K(a, c)$. We then consider the cube condition on triples (c, b, a), (d, c, b), and (b, g, f); constructing atoms $\{g := c \setminus d, h := c \setminus e, i := g \setminus c\}$. Then we have the atoms of G are $\{a, b, c, d, e, f, g, h, i\}$, satisfying cycles:

$$
\begin{aligned}
C_b =_h C &= K(d, c, g, i) \\
C_h =_c C &= K(a, b, d, e) \\
C_c =_b C &= K(a, f, g, h) \\
C_a =_g C &= K(b, c, f) \\
C_g =_d C &= K(h, b, i) \\
C_d =_a C &= K(e, c, h) \\
C_e =_f C &= K(a, c) \\
C_f =_i C &= K(b, g) \\
C_i =_e C &= K(d, h)
\end{aligned}
$$

and we can relabel the atoms to see that G is B_2.

This resolves all of the cases and completes the proof.

■

We have seen that for $k = 2, 3, 4$ there are only finitely many Bessis-Garside groups of rank 3 with all cycles of length at most k. The following counterexample demonstrates that this is not the case for all $k \in \mathbb{N}$.

Example 5.1.11 *There exists an infinite family of Bessis-Garside groups with finite sets of atoms and all cycles of length at most* 7*, namely:*

A^J

$A = \{\alpha_i^j, \beta_i, \gamma_i \mid i \in \mathbb{Z}_{3J+2}, j \in \{1, \dots, J\}\}$

$\alpha_i^1(\alpha_{i+2}^1, \alpha_{i-1}^2, \alpha_{i-3}^2, \alpha_{i-1}^1, \beta_{i-2}, \gamma_{i-1}, \beta_{i-1})$	α_{i+1}^1	
$\beta_i(\alpha_{i+3}^1, \alpha_{i+2}^1, \gamma_{i+1})$	β_{i+1}	
$\gamma_i(\alpha_{i+2}^1, \beta_i)$	γ_{i+1}	
$\alpha_i^j(\alpha_{i-j}^j, \alpha_{i+j}^{j-1}, \alpha_{i+2j}^{j-1}, \alpha_{i+j+1}^j, \alpha_{i-j}^{j+1}, \alpha_{i-2j-1}^{j+1})$	α_{i+1}^j	*for* $2 \leq j \leq J-1$
$\alpha_i^J(\alpha_{i-J}^J, \alpha_{i+J}^{J-1}, \alpha_{i+2J}^{J-1}, \alpha_{i+J+1}^J)$	α_{i+1}^J	

5.2 Bessis-Garside groups of uniform cycle length

Corollary 5.2.1 *For* $k = 2, 3, 4$ *there exists exactly one Bessis-Garside group with all cycles of length* k*, namely* $\mathbb{Z}_3$*,* $\tilde{A}_2(2)$*, and the group* P_{24}*.*

There are at least 3 Bessis-Garside groups of rank 3 with cycle presentations of the form given in Theorem 4.3.1 having all cycles of length 5.

Example 5.2.2 *We call* P_{120} *the cycle presented group with set of atoms* $A_{10} = \{a_s, b_s, \dots, l_s \mid s \in \mathbb{Z}_{10}\}$ *and cycles:*

$$
\begin{aligned}
C_{a_s} &=_{a_{s+1}} C = (b_s, c_s, f_{s+1}, g_{s+1}, i_s)\\
C_{b_s} &=_{b_{s+1}} C = (d_s, c_s, a_{s+1}, i_{s+1}, j_s)\\
C_{c_s} &=_{c_{s+1}} C = (a_{s+1}, b_{s+1}, d_{s+1}, e_{s+1}, f_{s+1})\\
C_{d_s} &=_{d_{s+1}} C = (e_s, c_s, b_{s+1}, j_{s+1}, l_{s+1})\\
C_{e_s} &=_{e_{s+1}} C = (h_{s-1}, f_s, c_s, d_{s+1}, l_{s+2})\\
C_{f_s} &=_{f_{s+1}} C = (a_s, c_s, e_{s+1}, h_s, g_s)\\
C_{g_s} &=_{g_{s+1}} C = (i_{s-1}, a_s, f_{s+1}, h_{s+1}, k_s)\\
C_{h_s} &=_{h_{s+1}} C = (k_{s-1}, g_s, f_{s+1}, e_{s+2}, l_{s+4})\\
C_{i_s} &=_{i_{s+1}} C = (j_{s-1}, b_s, a_{s+1}, g_{s+2}, k_{s+2})\\
C_{j_s} &=_{j_{s+1}} C = (d_s, b_{s+1}, i_{s+2}, k_{s+4}, l_s)\\
C_{k_s} &=_{k_{s+1}} C = (j_{s-3}, i_{s-1}, g_{s+1}, h_{s+2}, l_{s+6})\\
C_{l_s} &=_{l_{s+1}} C = (e_{s-1}, d_s, j_{s+1}, k_{s+5}, h_{s-3})
\end{aligned}
$$

P_{120} *is clearly Bessis-Garside of rank* 3 *by Theorem 4.3.1.*

If we consider the groups with the same cycles but with A_{10}*, replaced by* $A_2 = \{a_s, b_s, \ldots, l_s \mid s \in \mathbb{Z}_2\}$ *and* $A_5 = \{a_s, b_s, \ldots, l_s \mid s \in \mathbb{Z}_5\}$*, we have* 2 *more Bessis-Garside groups with all cycles of length* 5 *with* 24 *and* 60 *atoms respectively.*

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